CHAPTER 8 TECHNIQUES OF INTEGRATION

8.1 INTEGRATION BY PARTS

1. \( u = x, \ du = dx; \ dv = \sin \frac{x}{2} \ dx, \ v = -2 \cos \frac{x}{2} ; \)
   \[ \int x \sin \frac{x}{2} \ dx = -2x \cos \frac{x}{2} - \int \left( -2 \cos \frac{x}{2} \right) \ dx = -2x \cos \left( \frac{x}{2} \right) + 4 \sin \left( \frac{x}{2} \right) + C \]

2. \( u = \theta, \ du = d\theta; \ dv = \cos \pi \theta \ d\theta, \ v = \frac{1}{\pi} \sin \pi \theta; \)
   \[ \int \theta \cos \pi \theta \ d\theta = \left. \frac{\theta}{\pi} \sin \pi \theta \right| - \int \frac{1}{\pi} \sin \pi \theta \ d\theta = \left. \frac{\theta}{\pi} \sin \pi \theta + \frac{1}{\pi^2} \cos \pi \theta \right| + C \]

3. \( x^2 \ln x, \ du = \frac{dx}{x} ; \ dv = x \ dx, \ v = \frac{x^2}{2} ; \)
   \[ \int x^2 \ln x \ dx = \left[ \frac{x^2}{2} \ln x \right]_1^2 - \int_1^2 \frac{x^2}{2} \frac{dx}{x} = 2 \ln 2 - \left[ \frac{x^2}{4} \right]_1^2 = 2 \ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4} \]

4. \( x^2 \ln x, \ du = 2x \ dx; \ dv = \cos x \ dx, \ v = \sin x ; \)
   \[ \int x^2 \cos x \ dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C \]

5. \( u = \ln x, \ du = \frac{dx}{x} ; \ dv = x \ dx, \ v = \frac{x^2}{2} ; \)
   \[ \int_1^e x \ln x \ dx = \left[ \frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{x^2}{2} \frac{dx}{x} = e^2 - \left[ \frac{x^2}{4} \right]_1^e = e^2 - \frac{5}{4} \]

6. \( u = \ln x, \ du = \frac{dx}{x} ; \ dv = x^3 \ dx, \ v = \frac{x^4}{4} ; \)
   \[ \int_1^e x^3 \ln x \ dx = \left[ \frac{x^4}{4} \ln x \right]_1^e - \int_1^e \frac{x^4}{4} \frac{dx}{x} = e^4 - \left[ \frac{x^5}{20} \right]_1^e = \frac{3e^4 + 1}{16} \]

7. \( u = x, \ du = dx; \ dv = e^x \ dx, \ v = e^x ; \)
   \[ \int xe^x \ dx = xe^x - \int e^x \ dx = xe^x - e^x + C \]

8. \( u = x, \ du = dx; \ dv = e^{3x} \ dx, \ v = \frac{1}{3} e^{3x} ; \)
   \[ \int xe^{3x} \ dx = \frac{x}{3} e^{3x} - \frac{1}{9} \int e^{3x} \ dx = \frac{x}{3} e^{3x} - \frac{1}{9} e^{3x} + C \]
Chapter 8 Techniques of Integration

9. \[ e^{-x} \]
   \[ x^2 \rightarrow -e^{-x} \]
   \[ 2x \rightarrow e^{-x} \]
   \[ 2 \rightarrow -e^{-x} \]
   \[ \int x^2 e^{-x} \, dx = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C \]

10. \[ e^{2x} \]
    \[ x^2 - 2x + 1 \rightarrow \frac{1}{2} e^{2x} \]
    \[ 2x - 2 \rightarrow \frac{1}{4} e^{2x} \]
    \[ 2 \rightarrow \frac{1}{8} e^{2x} \]
    \[ \int (x^2 - 2x + 1) e^{2x} \, dx = \frac{1}{2}(x^2 - 2x + 1) e^{2x} - \frac{1}{4}(2x - 2) e^{2x} + \frac{1}{8} e^{2x} + C \]

11. \[ u = \tan^{-1} y, \ du = \frac{dy}{1+y^2}; \ dv = dy, \ v = y; \]
    \[ \int \tan^{-1} y \ dy = y \tan^{-1} y - \int \frac{y \ dy}{1+y^2} = y \tan^{-1} y - \frac{1}{2} \ln (1 + y^2) + C = y \tan^{-1} y - \ln \sqrt{1+y^2} + C \]

12. \[ u = \sin^{-1} y, \ du = \frac{dy}{\sqrt{1-y^2}}; \ dv = dy, \ v = y; \]
    \[ \int \sin^{-1} y \ dy = y \sin^{-1} y - \int \frac{y \ dy}{\sqrt{1-y^2}} = y \sin^{-1} y + \sqrt{1-y^2} + C \]

13. \[ u = x, \ du = dx; \ dv = \sec^2 x \, dx, \ v = \tan x; \]
    \[ \int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x + \ln |\cos x| + C \]

14. \[ \int 4x \sec^2 2x \, dx; [y = 2x] \rightarrow \int y \sec^2 y \ dy = y \tan y - \int \tan y \ dy = y \tan y - \ln |\sec y| + C \]
    \[ = 2x \tan 2x - \ln |\sec 2x| + C \]

15. \[ e^x \]
    \[ x^3 \rightarrow e^x \]
    \[ 3x^2 \rightarrow e^x \]
    \[ 6x \rightarrow e^x \]
    \[ 6 \rightarrow e^x \]
    \[ \int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6) e^x + C \]
16. \[ e^{-p} \]

\[ \begin{align*}
\text{p}^4 & \qquad (+) \qquad -e^{-p} \\
4p^3 & \qquad (-) \qquad e^{-p} \\
12p^2 & \qquad (+) \qquad -e^{-p} \\
24p & \qquad (-) \qquad e^{-p} \\
24 & \qquad (+) \qquad -e^{-p} \\
\end{align*} \]

\[ \int p^4 e^{-p} \, dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24pe^{-p} - 24e^{-p} + C \]
\[ = (-p^4 - 4p^3 - 12p^2 - 24p - 24) e^{-p} + C \]

17. \[ e^x \]

\[ \begin{align*}
x^2 - 5x & \qquad (+) \qquad e^x \\
2x - 5 & \qquad (-) \qquad e^x \\
2 & \qquad (+) \qquad e^x \\
\end{align*} \]

\[ \int (x^2 - 5x) e^x \, dx = (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x + C = x^2 e^x - 7xe^x + 7e^x + C \]
\[ = (x^2 - 7x + 7) e^x + C \]

18. \[ e^r \]

\[ \begin{align*}
r^2 + r + 1 & \qquad (+) \qquad e^r \\
2r + 1 & \qquad (-) \qquad e^r \\
2 & \qquad (+) \qquad e^r \\
\end{align*} \]

\[ \int (r^2 + r + 1) e^r \, dr = (r^2 + r + 1)e^r - (2r + 1)e^r + 2e^r + C \]
\[ = [(r^2 + r + 1) - (2r + 1) + 2] e^r + C = (r^2 - r + 2) e^r + C \]

19. \[ e^x \]

\[ \begin{align*}
x^5 & \qquad (+) \qquad e^x \\
5x^4 & \qquad (-) \qquad e^x \\
20x^3 & \qquad (+) \qquad e^x \\
60x^2 & \qquad (-) \qquad e^x \\
120x & \qquad (+) \qquad e^x \\
120 & \qquad (-) \qquad e^x \\
\end{align*} \]

\[ \int x^5 e^x \, dx = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe^x - 120e^x + C \]
\[ = (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) e^x + C \]
20. \[ e^t \rightarrow \frac{1}{2} e^u \]
\[ 2t \rightarrow \frac{1}{16} e^u \]
\[ 2 \rightarrow \frac{1}{64} e^u \]

\[
\int t^2 e^u \, dt = \frac{e^u}{4} - \frac{3}{16} e^u + \frac{2}{64} e^u + C = \frac{e^u}{4} - \frac{1}{8} e^u + \frac{1}{32} e^u + C
\]

\[ = \left( \frac{1}{4} - \frac{1}{8} + \frac{1}{32} \right) e^u + C \]

21. \[ I = \int e^u \sin \theta \, d\theta; \quad [u = \sin \theta, \, du = \cos \theta \, d\theta; \, dv = e^v \, d\theta, \, v = e^v] \Rightarrow I = e^u \sin \theta - \int e^v \cos \theta \, d\theta; \]
\[ [u = \cos \theta, \, du = -\sin \theta \, d\theta; \, dv = e^v \, d\theta, \, v = e^v] \Rightarrow I = e^u \sin \theta - \left( e^v \cos \theta + \int e^v \sin \theta \, d\theta \right) \]
\[ = e^u \sin \theta - e^v \cos \theta - I + C' \Rightarrow 2I = (e^u \sin \theta - e^v \cos \theta) + C' \Rightarrow I = \frac{1}{2} (e^u \sin \theta - e^v \cos \theta) + C, \text{ where } C = \frac{C'}{2} \text{ is another arbitrary constant} \]

22. \[ I = \int e^{-y} \cos y \, dy; \quad [u = \cos y, \, du = -\sin y \, dy; \, dv = e^{-y} \, dy, \, v = -e^{-y}] \]
\[ \Rightarrow I = -e^{-y} \cos y - \int (-e^{-y}) (-\sin y) \, dy = -e^{-y} \cos y - \int e^{-y} \sin y \, dy; \quad [u = \sin y, \, du = \cos y \, dy; \, dv = -e^{-y} \, dy, \, v = -e^{-y}] \]
\[ \Rightarrow I = -e^{-y} \cos y - \left( -e^{-y} \sin y - \int (-e^{-y}) \cos y \, dy \right) = -e^{-y} \cos y + e^{-y} \sin y - I + C' \]
\[ \Rightarrow 2I = e^{-y} (\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2} (e^{-y} \sin y - e^{-y} \cos y) + C, \text{ where } C = \frac{C'}{2} \]

23. \[ I = \int e^{2u} \cos 3x \, dx; \quad [u = \cos 3x; \, du = -3 \sin 3x \, dx, \, dv = e^{2u} \, dx, \, v = \frac{1}{4} e^{2u}] \]
\[ \Rightarrow I = \frac{1}{2} e^{2u} \cos 3x + \frac{3}{2} \int e^{2u} \sin 3x \, dx; \quad [u = \sin 3x, \, du = 3 \cos 3x, \, dv = e^{2u} \, dx, \, v = \frac{1}{4} e^{2u}] \]
\[ \Rightarrow I = \frac{1}{2} e^{2u} \cos 3x + \frac{3}{2} \left( \frac{1}{2} e^{2u} \sin 3x - \frac{3}{2} \int e^{2u} \cos 3x \, dx \right) = \frac{3}{4} e^{2u} \cos 3x + \frac{3}{4} e^{2u} \sin 3x - \frac{9}{4} I + C' \]
\[ \Rightarrow \frac{12}{4} I = \frac{3}{4} e^{2u} \cos 3x + \frac{3}{4} e^{2u} \sin 3x + C' \Rightarrow I = \frac{1}{4} \left( 3 \sin 3x + 2 \cos 3x \right) + C, \text{ where } C = \frac{C'}{4} + C' \]

24. \[ I = \int e^{-2u} \sin 2x \, dx; \quad [y = 2x] \rightarrow \frac{1}{2} \int e^{-y} \sin y \, dy = I; \quad [u = \sin y, \, du = \cos y \, dy; \, dv = e^{-y} \, dy, \, v = -e^{-y}] \]
\[ \Rightarrow I = -\frac{1}{2} e^{-y} \sin y + \int e^{-y} \cos y \, dy \quad [u = \cos y, \, du = -\sin y \, dy; \, dv = e^{-y} \, dy, \, v = -e^{-y}] \]
\[ \Rightarrow I = -\frac{1}{2} e^{-y} \sin y + \frac{1}{2} (-e^{-y} \cos y - \int (-e^{-y}) (-\sin y) \, dy) = -\frac{1}{2} e^{-y} (\sin y + \cos y) - I + C' \]
\[ \Rightarrow 2I = -\frac{1}{2} e^{-y} (\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{2} e^{-y} (\sin y + \cos y) + C = -\frac{e^{-y}}{4} (\sin 2x + \cos 2x) + C, \text{ where } C = \frac{C'}{2} \]

25. \[ I = \int e^{\sqrt{3u}} \, ds; \quad \left[ \frac{3s + 9 = x^2}{ds = \frac{x}{3} \, dx} \right] \rightarrow \int e^{x} \cdot \frac{x}{3} \, dx = \frac{2}{3} \int e^{x} \, dx; \quad [u = x, \, du = dx; \, dv = e^{x} \, dx, \, v = e^{x}] \]
\[ \frac{2}{3} \int e^{x} \, dx = \frac{2}{3} (e^{x} - e^{x}) + C = \frac{2}{3} \left( \sqrt{3s + 9} e^{\sqrt{3u}} - e^{\sqrt{3u}} \right) + C \]

26. \[ u = x, \, du = dx; \quad dv = \sqrt{1-x} \, dx, \, v = -\frac{1}{2} \sqrt{(1-x)^3}; \quad \int_{0}^{1} \sqrt{1-x} \, dx = \left[ -\frac{1}{2} \sqrt{(1-x)^3} \right]_{0}^{1} + \frac{1}{2} \int_{0}^{1} \sqrt{(1-x)^3} \, dx = \frac{3}{2} \left[ -\frac{1}{2} (1-x)^{3/2} \right]_{0}^{1} = \frac{1}{12} \]

27. \[ u = x, \, du = dx; \quad dv = \tan^2 x \, dx, \quad v = \int \tan^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1-\cos^2 x}{\cos^2 x} \, dx = \int \frac{dx}{\cos^2 x} - \int \frac{\cos^2 x}{\cos^2 x} \, dx \]
\[ = \tan x - x; \quad \int_{0}^{\pi/3} \tan^2 x \, dx = \int_{0}^{\pi/3} (\tan x - x) \, dx = \frac{\pi}{2} \left( \sqrt{3} - \frac{\pi}{3} \right) + \left[ \ln |\cos x| + \frac{x^2}{2} \right]_{0}^{\pi/3} \]
\[ = \frac{\pi}{2} \left( \sqrt{3} - \frac{\pi}{3} \right) + \ln \frac{\sqrt{3}}{2} - \ln 2 - \frac{\pi^2}{18} \]
28. \( u = \ln(x + x^2) \), \( du = \frac{(2x + 1)dx}{x + x^2} \); \( dv = dx \), \( v = x \); \( \int \ln(x + x^2) \, dx = x \ln(x + x^2) - \int \frac{2x + 1}{x + x^2} \cdot x \, dx \)

\[ = x \ln(x + x^2) - \int \frac{2x + 1}{x + x^2} \, dx = x \ln(x + x^2) - 2x + \ln|x + 1| + C \]

29. \( \int \sin(\ln x) \, dx; \) \( du = \frac{1}{x} \, dx \), \( v = \ln x \); \( \int \sin(\ln x) \, dx = \ln x \cos(\ln x) + C \)

\[ = \frac{1}{2} \left[ -x \cos(\ln x) + x \sin(\ln x) \right] + C \]

30. \( \int (\ln z)^2 \, dz; \) \( \int \left[ u = \ln z \right] \frac{1}{z} \, dz \)

\[ \Rightarrow \int e^u \cdot u^2 \cdot e^u \, du = \int e^{2u} \cdot u^2 \, du; \]

\[ \begin{align*}
    u^2 & \rightarrow \frac{1}{2} e^{2u} \\
    2u & \rightarrow \frac{1}{4} e^{2u} \\
    2 & \rightarrow \frac{1}{8} e^{2u}
\end{align*} \]

\[ \int u^2 e^{2u} \, du = \frac{u^2}{2} e^{2u} - \frac{u}{2} e^{2u} + \frac{1}{4} e^{2u} + C = \frac{e^u}{4} [2u^2 - 2u + 1] + C \]

\[ = \frac{e^u}{4} [2(\ln z)^2 - 2 \ln z + 1] + C \]

31. \( \int x \sec^2 x \, dx \) \( \Rightarrow \) \( u = x^2 \), \( du = 2x \, dx \)

\[ = \frac{1}{2} \int \sec^2 x \, dx = \frac{1}{2} \int \sec u \, du = \frac{1}{2} \ln|\sec u + \tan u| + C \]

\[ = \frac{1}{2} \ln|\sec x^2 + \tan x^2| + C \]

32. \( \int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx \)

\[ \Rightarrow \] \( u = \sqrt{x} \), \( du = \frac{1}{2 \sqrt{x}} \, dx \)

\[ = 2 \int \cos u \, du = 2 \sin u + C = 2 \sin \sqrt{x} + C \]

33. \( \int x(\ln x)^2 \, dx; \) \( \int \left[ u = \ln x \right] \frac{1}{x} \, dx \)

\[ \Rightarrow \int e^u \cdot u^2 \cdot e^u \, du = \int e^{2u} \cdot u^2 \, du; \]

\[ \begin{align*}
    u^2 & \rightarrow \frac{1}{2} e^{2u} \\
    2u & \rightarrow \frac{1}{4} e^{2u} \\
    2 & \rightarrow \frac{1}{8} e^{2u}
\end{align*} \]

\[ \int u^2 e^{2u} \, du = \frac{u^2}{2} e^{2u} - \frac{u}{2} e^{2u} + \frac{1}{4} e^{2u} + C = \frac{e^u}{4} [2u^2 - 2u + 1] + C \]

\[ = \frac{e^u}{4} [2(\ln x)^2 - 2 \ln x + 1] + C \]

34. \( \int \frac{1}{x(\ln x)^2} \, dx \)

\[ \Rightarrow \left[ u = \ln x \right] \frac{1}{x} \, du = \frac{1}{x} \, du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C \]

35. \( \int \frac{u}{x} \, dx \) \( \Rightarrow \) \( du = \frac{1}{x} \, dx \), \( v = -\frac{1}{x} \);

\[ \int \frac{\ln x}{x^2} \, dx = -\frac{\ln x}{x} + \int \frac{1}{x} \, dx = -\frac{\ln x}{x} - \frac{1}{x} + C \]

36. \( \int \frac{u^3}{x} \, dx \)

\[ \Rightarrow \] \( u = \ln x \), \( du = \frac{1}{x} \, dx \)

\[ = \int \frac{(\ln x)^3}{x} \, dx = \int u^3 \, du = \frac{1}{4} u^4 + C = \frac{1}{4} (\ln x)^4 + C \]
37. \( \int x^3 e^x \, dx \) Let \( u = x^3 \), \( du = 3x^2 \, dx \) \( \Rightarrow \frac{1}{3} du = x^2 \, dx \) \( \rightarrow \int x^3 e^x \, dx = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C = \frac{1}{3} e^x + C \\
\)

38. \( u = x^3 \), \( du = 3x^2 \, dx \); \( dv = e^x \, dx \), \( v = e^x \) \( \Rightarrow \)
\( \int x^3 e^x \, dx = \frac{1}{3} x^3 e^x - \frac{1}{3} \int 3x^2 e^x \, dx = \frac{1}{3} x^3 e^x - \frac{1}{3} e^x + C \)
\( \)

39. \( u = x^2 \), \( du = 2x \, dx \); \( dv = \sqrt{x^2 + 1} \, dx \), \( v = \frac{1}{2} (x^2 + 1)^{3/2} \) \( \Rightarrow \)
\( \int x^2 \sqrt{x^2 + 1} \, dx = \frac{1}{2} x^2 (x^2 + 1)^{3/2} - \frac{1}{2} \int (x^2 + 1)^{3/2} \, 2x \, dx = \frac{1}{3} x^2 (x^2 + 1)^{3/2} - \frac{2}{15} (x^2 + 1)^{5/2} + C \)
\( \)

40. \( \int x^3 \sin x \, dx \) Let \( u = x^3 \), \( du = 3x^2 \, dx \) \( \Rightarrow \frac{1}{3} du = x^2 \, dx \) \( \rightarrow \int x^3 \sin x \, dx = \frac{1}{3} \int \sin u \, du = -\frac{1}{3} \cos u + C = -\frac{1}{3} \cos x^3 + C \)
\( \)

41. \( \int \sin 3x \, dx \) \( \Rightarrow \frac{1}{3} \sin 3x \, dx \) \( \Rightarrow \frac{1}{3} \sin 2x \) \( \Rightarrow \)
\( \int \sin 3x \, dx = -\frac{1}{3} \sin 3x \, dx = -\frac{1}{3} \sin 2x \) \( \Rightarrow \)
\( \int \sin 3x \, dx = -\frac{1}{3} \sin 3x \, dx = -\frac{1}{3} \sin 2x \)
\( \)

42. \( \int \cos 3x \, dx \) \( \Rightarrow \)
\( \int \cos 3x \, dx = \frac{1}{3} \cos 3x \, dx = \frac{1}{3} \cos 2x \) \( \Rightarrow \)
\( \int \cos 3x \, dx = \frac{1}{3} \cos 3x \, dx = \frac{1}{3} \cos 2x \)
\( \)

43. \( \int e^x \sin e^x \, dx \) \( \Rightarrow \int e^x \sin e^x \, dx \) \( \Rightarrow \int e^u \, du = \int -\cos u + C = -\cos e^x + C \)
\( \)

44. \( \int \sqrt{x} \, dx \) \( \Rightarrow \frac{1}{2} \sqrt{x} \, dx \) \( \Rightarrow \frac{1}{2} \sqrt{x} \) \( \Rightarrow \)
\( \int \sqrt{x} \, dx = \frac{1}{2} \sqrt{x} \, dx = \frac{1}{2} \sqrt{x} \)
\( \)

45. \( \int \cos \sqrt{x} \, dx \) \( \Rightarrow \int \cos \sqrt{x} \, dx \) \( \Rightarrow \int \cos y \, dy = \int 2y \, cos y \, dy \)
\( \Rightarrow \)
\( u = 2y \), \( du = 2 \, dy \); \( dv = \cos y \, dy \), \( v = \sin y \)
\( \Rightarrow \)
\( \int 2y \, cos y \, dy = 2y \sin y \) \( \Rightarrow \int 2y \, sin y \, dy = 2y \sin y + 2 \cos y + C = 2 \sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C \)

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46. \[\int \sqrt{x} e^{\sqrt{x}} \, dx; \quad y = \sqrt{x} \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \int y \, e^y \, dy = \int 2y^2 \, e^y \, dy;\]
\[
\begin{align*}
2y^2 & \quad (+) \quad e^y \\
4y & \quad (-) \quad e^y \\
4 & \quad (+) \quad e^y \\
0 & \quad \int 2y^2 \, e^y \, dy = 2y^2 e^y - 4y e^y + 4 e^y + C = 2x e^{\sqrt{x}} - 4\sqrt{x} e^{\sqrt{x}} + 4e^{\sqrt{x}} + C
\end{align*}
\]

47. \[
\sin 2\theta \quad \theta^2 \quad (+) \quad \cos 2\theta
\]
\[
\begin{align*}
\int_{\theta_0}^{\theta_1} \theta^2 \sin 2\theta \, d\theta &= \left[ -\frac{\theta^3}{3} \cos 2\theta + \frac{\theta}{2} \sin 2\theta + \frac{1}{2} \cos 2\theta \right]_{\theta_0}^{\theta_1} \\
&= \left[ -\frac{\pi^3}{18} \cdot (-1) + \frac{\pi}{4} \cdot 0 + \frac{1}{2} \cdot (-1) \right] - \left[ 0 + 0 + \frac{1}{2} \cdot 1 \right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2-4}{8}
\end{align*}
\]

48. \[
\cos 2x \quad x^3 \quad (+) \quad \frac{1}{2} \sin 2x
\]
\[
\begin{align*}
x^3 & \quad (-) \quad -\frac{1}{4} \cos 2x \\
6x & \quad (+) \quad -\frac{1}{8} \sin 2x \\
6 & \quad (-) \quad \frac{1}{16} \cos 2x \\
0 & \quad \int_{0}^{\theta_1/2} x^3 \cos 2x \, dx = \left[ \frac{x^4}{4} \sin 2x + \frac{3x^3}{3} \cos 2x - \frac{3x^2}{4} \sin 2x - \frac{x}{8} \cos 2x \right]_{0}^{\theta_1/2} \\
&= \left[ \frac{\pi^4}{16} \cdot 0 + \frac{3\pi^3}{16} \cdot (-1) - \frac{3\pi^2}{8} \cdot 0 - \frac{\pi}{8} \cdot (-1) \right] - \left[ 0 + 0 - \frac{3\pi}{8} \cdot 1 \right] = -\frac{3\pi^3}{16} + \frac{\pi}{8} = \frac{3(4-\pi^2)}{16}
\end{align*}
\]

49. \[
u = \sec^{-1} t, \quad du = \frac{dt}{t\sqrt{t^2-1}}; \quad dv = t \, dt, \quad v = \frac{1}{2};
\]
\[
\int_{\sqrt{3}}^{2/\sqrt{3}} t \sec^{-1} t \, dt = \left[ \frac{\pi}{2} \sec^{-1} t \right]_{\sqrt{3}}^{2/\sqrt{3}} \quad - \frac{1}{2} \int_{\sqrt{3}}^{2/\sqrt{3}} \frac{dt}{t\sqrt{t^2-1}} = \left( \int_{\sqrt{3}}^{2/\sqrt{3}} \frac{dt}{t\sqrt{t^2-1}} \right) - \frac{1}{2} \int_{\sqrt{3}}^{2/\sqrt{3}} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2} - \frac{1}{2} \left( \sqrt{3} - \sqrt{1} \right) = \frac{\pi}{2} - \frac{1}{2} \left( \sqrt{3} - \sqrt{3} \right) = \frac{\pi}{2} - \frac{\sqrt{3}}{2} = \frac{5\pi-3\sqrt{3}}{4}
\]

50. \[
u = \sin^{-1} (x^2), \quad du = \frac{2x \, dx}{\sqrt{1-x^4}}; \quad dv = 2x \, dx, \quad v = x^2;
\]
\[
\int_{0}^{1/\sqrt{2}} 2x \sin^{-1} (x^2) \, dx = \left[ x^2 \sin^{-1} (x^2) \right]_{0}^{1/\sqrt{2}} - \int_{0}^{1/\sqrt{2}} x^2 \cdot \frac{2x \, dx}{\sqrt{1-x^4}} = \left( \frac{1}{2} \right) \left( \frac{\pi}{2} \right) + \int_{0}^{1/\sqrt{2}} \frac{2x \, dx}{\sqrt{1-x^4}} = \frac{\pi}{12} + \sqrt{\frac{\pi}{4}} - 1 = \frac{\pi+6\sqrt{3}-12}{12}
\]

51. (a) \[u = x, \quad du = dx; \quad dv = \sin x \, dx, \quad v = -\cos x;
\]
\[
S_1 = \int_{0}^{\pi} \sin x \, dx = [-x \cos x]_{0}^{\pi} + \int_{0}^{\pi} \cos x \, dx = \pi + [\sin x]_{0}^{\pi} = \pi
\]
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(b) \[ S_2 = - \int_0^{\pi/2} x \sin x \, dx = - \left[ -x \cos x \right]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx = - \left[ -3\pi + [\sin x]_0^{\pi/2} \right] = 3\pi \]

(c) \[ S_3 = \int_0^{3\pi/2} x \sin x \, dx = - \left[ -x \cos x \right]_0^{3\pi/2} + \int_0^{3\pi/2} \cos x \, dx = 5\pi + [\sin x]_0^{3\pi/2} = 5\pi \]

(d) \[ S_{n+1} = (-1)^{n+1} \int_0^{(n+1)\pi/2} x \sin x \, dx = (-1)^{n+1} \left[ -x \cos x \right]_0^{(n+1)\pi/2} + [\sin x]_0^{(n+1)\pi/2} \]

52. (a) \[ u = x, \ du = dx; \ dv = \cos x \, dx, \ v = \sin x; \]

\[ S_1 = - \int_{\pi/2}^{\pi/2} x \cos x \, dx = - \left[ x \sin x \right]_{\pi/2}^{\pi/2} - \int_{\pi/2}^{\pi/2} \sin x \, dx = - \left[ -\frac{3\pi}{2} - \frac{\pi}{2} \right] = \cos x |_{\pi/2}^{\pi/2} = 2\pi \]

(b) \[ S_2 = \int_{\pi/2}^{3\pi/2} x \cos x \, dx = \left[ x \sin x \right]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x \, dx = \left[ 5\pi \right] - \left[ -\frac{3\pi}{2} - \frac{\pi}{2} \right] = \cos x |_{\pi/2}^{3\pi/2} = 4\pi \]

(c) \[ S_3 = - \int_{\pi/2}^{5\pi/2} x \cos x \, dx = - \left[ x \sin x \right]_{\pi/2}^{5\pi/2} - \int_{\pi/2}^{5\pi/2} \sin x \, dx = - \left[ -\frac{7\pi}{2} - \frac{5\pi}{2} \right] = \cos x |_{\pi/2}^{5\pi/2} = 6\pi \]

(d) \[ S_n = (-1)^n \int_{n\pi/2}^{(n+1)\pi/2} x \cos x \, dx = (-1)^n \left[ x \sin x \right]_{n\pi/2}^{(n+1)\pi/2} - \int_{n\pi/2}^{(n+1)\pi/2} \sin x \, dx \]

53. \[ V = \int_0^{\ln 2} 2(\ln 2 - x)e^x \, dx = 2\pi \ln 2 \int_0^{\ln 2} e^x \, dx - 2\pi \int_0^{\ln 2} xe^x \, dx \]

\[ = (2\pi \ln 2) \left[ e^x \right]_0^{\ln 2} - 2\pi \left[ xe^x \right]_0^{\ln 2} \int_0^{\ln 2} e^x \, dx \]

\[ = 2\pi \ln 2 - 2\pi \left( 2 \ln 2 - [-e^{-1}]_0^{\ln 2} \right) = -2\pi \ln 2 + 2\pi = 2\pi(1 - \ln 2) \]

54. (a) \[ V = \int_0^1 2\pi xe^{-x} \, dx = 2\pi \left[ xe^{-x} \right]_0^1 + \int_1^1 e^{-x} \, dx \]

\[ = 2\pi \left( -\frac{1}{e} + [-e^{-1}]_0^1 \right) = 2\pi \left( -\frac{1}{e} - \frac{1}{e} + 1 \right) \]

\[ = 2\pi - \frac{4\pi}{e} \]

(b) \[ V = \int_0^1 2\pi(1 - x)e^{-x} \, dx; \quad u = 1 - x, \ du = -dx; \ dv = e^{-x} \, dx, \]

\[ v = -e^{-x}; \quad V = 2\pi \left[ (1 - x)(-e^{-x}) \right]_0^1 - \int_0^1 e^{-x} \, dx \]

\[ = 2\pi \left( 0 - (-1) \right) + [e^{-x}]_0^1 = 2\pi \left( 1 + \frac{1}{e} - 1 \right) = \frac{2\pi}{e} \]

55. (a) \[ V = \int_0^{\pi/2} 2\pi x \cos x \, dx = 2\pi \left[ x \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} \cos x \, dx \]

\[ = 2\pi \left( \frac{\pi}{2} + [\cos x]_0^{\pi/2} \right) = 2\pi \left( \frac{\pi}{2} + 0 - 1 \right) = \pi(\pi - 2) \]

(b) \[ V = \int_0^{\pi/2} 2\pi \left( \frac{\pi}{2} - x \right) \cos x \, dx; \quad u = \frac{\pi}{2} - x, \ du = -dx; \ dv = \cos x \, dx, \ v = \sin x; \]

\[ V = 2\pi \left[ \left( \frac{\pi}{2} - x \right) \sin x \right]_0^{\pi/2} + 2\pi \int_0^{\pi/2} \sin x \, dx = 0 + 2\pi[-\cos x]_0^{\pi/2} = 2\pi(0 + 1) = 2\pi \]
56. (a) \[ V = \int_0^\pi 2\pi x(x \sin x) \, dx; \]
\[
\begin{align*}
    x^2 &\xrightarrow{(+)} -\cos x \\
    2x &\xrightarrow{(-)} -\sin x \\
    2 &\xrightarrow{(+) \cos x}
\end{align*}
\]
\[
0 \Rightarrow V = 2\pi \int_0^\pi x^2 \sin x \, dx = 2\pi \left[ -x^2 \cos x + 2x \sin x + 2 \cos x \right]_0^\pi = 2\pi (\pi^2 - 4)
\]
(b) \[ V = \int_0^\pi 2\pi(x - x)\sin x \, dx = 2\pi^2 \int_0^\pi \sin x \, dx - 2\pi \int_0^\pi x^2 \sin x \, dx = 2\pi^2 \left[ -x \cos x + \sin x \right]_0^\pi - (2\pi^3 - 8\pi) = 8\pi
\]

57. (a) \[ A = \int_1^e \ln x \, dx = \left[ x \ln x \right]_1^e - \int_1^e \, dx = (e \ln e - 1 \ln 1) - \left[ x \right]_1^e = e - (e - 1) = 1
\]
(b) \[ V = \int_1^e \pi (\ln x)^2 \, dx = \pi \left( \left[ x (\ln x)^2 \right]_1^e - \int_1^e 2 \ln x \, dx \right) = \pi \left( e(\ln e)^2 - 1(\ln 1)^2 \right) - \left( \left[ 2x \ln x \right]_1^e - \int_1^e 2 \, dx \right) = \pi \left( e^2 - (2e \ln e - 2(1) \ln 1) \right) - \left( 2x \right)_1^e = \pi \left( e^2 - (2e - 2) \right) = \pi(e - 2)
\]
(c) \[ V = \int_1^e 2\pi(x + 2) \ln x \, dx = 2\pi \int_1^e (x + 2) \ln x \, dx = 2\pi \left( \left[ \frac{1}{2}x^2 \ln x \right]_1^e - \int_1^e \left( \frac{1}{2}x + 2 \right) \, dx \right) = 2\pi \left( \left( \frac{1}{2}e^2 + 2e \right) \ln e - \left( \frac{1}{2} + 2 \right) \ln 1 - \left( \frac{1}{2}x^2 + 2x \right) \right)_1^e = 2\pi \left( \left( \frac{1}{2}e^2 + 2e \right) - 2 \right) = \frac{\pi}{2}(e^2 + 9)
\]
(d) \[ M = \int_1^e \ln x \, dx = 1 \text{ (from part (a)); } \]
\[
\begin{align*}
    x &\xrightarrow{1} \int_1^e x \ln x \, dx = \left[ \frac{1}{2}x^2 \ln x \right]_1^e - \int_1^e \frac{1}{2}x \, dx = \left( \frac{1}{2}e^2 \ln e - \frac{1}{2}(1)^2 \ln 1 \right) - \left( \frac{1}{2}x^2 \right)_1^e = \frac{1}{2}e^2 - \left( \frac{1}{2}e^2 - 2(1)^2 \right) = \frac{1}{2}(e^2 + 1); \]
    \[ y = \int_1^e (\ln x)^2 \, dx = \frac{1}{2} \left( \left[ x (\ln x)^2 \right]_1^e - \int_1^e 2 \ln x \, dx \right) = \frac{1}{2} \left( e^2 - 2 \right) \]
\]
\[
\begin{align*}
    &\Rightarrow (x, y) = \left( \frac{\pi + 1}{4}, \frac{\pi^2}{2} \right) \text{ is the centroid.}
\end{align*}
\]

58. (a) \[ A = \int_0^\pi \tan^{-1} x \, dx = \left[ x \tan^{-1} x \right]_0^\pi - \int_0^\pi \frac{x}{1 + x^2} \, dx
\]
\[
\begin{align*}
    &\Rightarrow A = (\tan^{-1} \pi - 0) - \frac{1}{2} \left[ \ln(1 + x^2) \right]_0^\pi = \frac{\pi}{4} - \frac{1}{2} \ln 2
\end{align*}
\]
(b) \[ V = \int_0^\pi 2\pi x \tan^{-1} x \, dx
\]
\[
\begin{align*}
    &\Rightarrow V = 2\pi \left( \left[ \frac{x^2}{2} \tan^{-1} x \right]_0^\pi - \frac{1}{2} \int_0^\pi \frac{x^2}{1 + x^2} \, dx \right) = 2\pi \left( \frac{x}{2} \tan^{-1} x \right)_0^\pi = 2\pi \left( \frac{\pi}{8} - \frac{1}{2} \ln 2 \right) = 2\pi \left( \frac{\pi - 2}{4} \right)
\end{align*}
\]
59. \( av(y) = \frac{1}{\pi} \int_0^{2\pi} 2e^{-t} \cos t \, dt \)
\[ = \frac{1}{\pi} \left[ e^{-t} \left( \sin \frac{t - \cos t}{2} \right) \right]_0^{2\pi} \]
(see Exercise 22) \( \Rightarrow av(y) = \frac{1}{\pi} \left( 1 - e^{-2\pi} \right) \)

60. \( av(y) = \frac{1}{\pi} \int_0^{2\pi} 4e^{-t} (\sin t - \cos t) \, dt \)
\[ = \frac{2}{\pi} \int_0^{2\pi} e^{-t} \sin t \, dt - \frac{2}{\pi} \int_0^{2\pi} e^{-t} \cos t \, dt \]
\[ = \frac{2}{\pi} \left[ e^{-t} \left( -\frac{\sin t - \cos t}{2} \right) - e^{-t} \left( \frac{\sin t + \cos t}{2} \right) \right]_0^{2\pi} \]
\[ = \frac{2}{\pi} \left[ -e^{-t} \sin t \right]_0^{2\pi} = 0 \]

61. \( I = \int x^2 \cos x \, dx; \, [u = x^n, \, du = nx^{n-1} \, dx; \, dv = \cos x \, dx, \, v = \sin x] \)
\[ \Rightarrow I = x^2 \sin x - \int nx^{n-1} \sin x \, dx \]

62. \( I = \int x^n \sin x \, dx; \, [u = x^n, \, du = nx^{n-1} \, dx; \, dv = \sin x \, dx, \, v = -\cos x] \)
\[ \Rightarrow I = -x^n \cos x + \int nx^{n-1} \cos x \, dx \]

63. \( I = \int x^a e^x \, dx; \, [u = x^n, \, du = nx^{n-1} \, dx; \, dv = e^x \, dx, \, v = \frac{1}{a} e^x] \)
\[ \Rightarrow I = \frac{x^a}{a} e^x - \frac{1}{a} \int x^{n-1} e^x \, dx, \, a \neq 0 \]

64. \( I = \int (\ln x)^n \, dx; \, [u = (\ln x)^n, \, du = \frac{n(\ln x)^{n-1} \, dx; \, dv = 1 \, dx, \, v = x] \)
\[ \Rightarrow I = x(\ln x)^n - \int n(\ln x)^{n-1} \, dx \]

65. \( I_a^b \int f(x) \, dx; \, [u = x - a, \, du = dx; \, dv = f(x) \, dx, \, v = \int_a^b f(t) \, dt = -\int_a^b f(t) \, dt] \)
\[ \Rightarrow I_a^b \left( f(x) \int_a^b f(t) \, dt \right) dx = \left( b - a \right) \int_a^b f(t) \, dt - \int_a^b \left( a - a \right) f(t) \, dt \]
\[ + 0 + \int_a^b \left( \int_a^b f(t) \, dt \right) dx = \int_a^b \left( \int_a^b f(t) \, dt \right) dx \]

66. \( I = \int 1 - x^2 \, dx; \, [u = x - a, \, du = dx; \, dv = dx, \, v = x] \)
\[ \Rightarrow x \sqrt{1 - x^2} - \int \frac{x^2}{\sqrt{1 - x^2}} \, dx = x \sqrt{1 - x^2} - \int \frac{x^2}{\sqrt{1 - x^2}} \, dx = x \sqrt{1 - x^2} - \left( \int \frac{x}{\sqrt{1 - x^2}} \, dx \right) \]
\[ = x \sqrt{1 - x^2} - \int \sqrt{1 - x^2} \, dx + \int \frac{1}{\sqrt{1 - x^2}} \, dx \]
\[ \Rightarrow x \sqrt{1 - x^2} = x \sqrt{1 - x^2} + \int \frac{1}{\sqrt{1 - x^2}} \, dx - \int \sqrt{1 - x^2} \, dx \Rightarrow 2 \int \sqrt{1 - x^2} \, dx = x \sqrt{1 - x^2} + \int \frac{1}{\sqrt{1 - x^2}} \, dx \]
\[ \Rightarrow \int \sqrt{1 - x^2} \, dx = \frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1 - x^2}} \, dx + C \]

67. \( I = \int \sin^{-1} x \, dx = x \sin^{-1} x - \int \sin y \, dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \cos (\sin^{-1} x) + C \)

68. \( I = \int \tan^{-1} x \, dx = x \tan^{-1} x - \int \tan y \, dy = x \tan^{-1} x + \ln |\cos y| + C = x \tan^{-1} x + \ln |\cos (\tan^{-1} x)| + C \)
\[ 69. \int \sec^{-1} x \, dx = x \sec^{-1} x - \int \sec y \, dy = x \sec^{-1} x - \ln |\sec y + \tan y| + C \\
= x \sec^{-1} x - \ln |\sec (\sec^{-1} x) + \tan (\sec^{-1} x)| + C \\
= x \sec^{-1} x - \ln \left| x + \sqrt{x^2 - 1} \right| + C \\
\]
\[ 70. \int \log_2 x \, dx = x \log_2 x - \int 2^y \, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{x}{\ln 2} + C \]

71. Yes, \( \cos^{-1} x \) is the angle whose cosine is \( x \) which implies \( \sin (\cos^{-1} x) = \sqrt{1-x^2} \).

72. Yes, \( \tan^{-1} x \) is the angle whose tangent is \( x \) which implies \( \sec (\tan^{-1} x) = \sqrt{1 + x^2} \).

73. (a) \( \int \sinh^{-1} x \, dx = x \sinh^{-1} x - \int \sinh y \, dy = x \sinh^{-1} x - \cosh (\sinh^{-1} x) + C \)
    
    \[
    \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x \, dx \\
    = \sinh^{-1} x \, dx \\
    \]

    (b) \( \int \sinh^{-1} x \, dx = x \sinh^{-1} x - \int x \left( \frac{1}{\sqrt{1+x^2}} \right) \, dx = x \sinh^{-1} x - 1/2 \int (1 + x^2)^{-1/2} \, dx \\
    = x \sinh^{-1} x - (1 + x^2)^{1/2} + C \)
    
    \[
    \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x \, dx \\
    \]

74. (a) \( \int \tanh^{-1} x \, dx = x \tanh^{-1} x - \int \tanh y \, dy = x \tanh^{-1} x - \ln |\cosh y| + C = x \tanh^{-1} x - \ln |\cosh (\tanh^{-1} x)| + C \)
    
    \[
    \frac{dx}{1-x^2} = \tanh^{-1} x \, dx \\
    = \tanh^{-1} x \, dx \\
    \]

    (b) \( \int \tanh^{-1} x \, dx = x \tanh^{-1} x - \int \frac{x}{1-x^2} \, dx = x \tanh^{-1} x - 1/2 \int \frac{2x}{1-x^2} \, dx = x \tanh^{-1} x + 1/2 \ln |1-x^2| + C \)
    
    \[
    \frac{dx}{1-x^2} = \tanh^{-1} x \, dx \\
    \]

\section*{8.2 Trigonometric Integrals}

1. \( \int \cos 2x \, dx = \frac{1}{2} \int \cos 2x \cdot 2 \, dx = \sin 2x + C \)

2. \( \int_0^\pi 3 \sin \frac{x}{2} \, dx = 9 \int_0^\pi \sin \frac{x}{2} \cdot \frac{1}{2} \, dx = 9 \left[ -\cos \frac{x}{2} \right]_0^\pi = 9(-\cos \frac{\pi}{2} + \cos 0) = 9(-\frac{1}{2} + 1) = \frac{9}{2} \)

3. \( \int \cos^3 x \sin x \, dx = -\int \cos^3 x (-\sin x) \, dx = -\frac{1}{4} \cos^4 x + C \)

4. \( \int \sin^4 2x \cos 2x \, dx = \frac{1}{2} \int \sin^4 2x \cos 2x \cdot 2 \, dx = \frac{1}{10} \sin^5 2x + C \)

5. \( \int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx = \int \sin x \, dx - \int \cos^2 x \sin x \, dx = -\cos x + \frac{1}{3} \cos^3 x + C \)

6. \( \int \cos^3 4x \, dx = \int \cos^2 4x \cos 4x \, dx = \frac{1}{4} \int (1 - \sin^2 4x) \cos 4x \cdot 4 \, dx = \frac{1}{4} \int \cos 4x \cdot 4 \, dx - \frac{1}{4} \int \sin^2 4x \cos 4x \cdot 4 \, dx \\
= \frac{1}{4} \sin 4x - \frac{1}{12} \sin^3 4x + C \)

7. \( \int \sin^3 x \, dx = \int (\sin^2 x)^2 \sin x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx = \int (1 - 2\cos^2 x + \cos^4 x) \sin x \, dx \\
= \int \sin x \, dx - \int 2\cos^2 x \sin x \, dx + \int \cos^4 x \sin x \, dx = -\cos x + \frac{3}{2} \cos^3 x - \frac{1}{3} \cos^5 x + C \)
8. \( \int_0^\pi \sin^5(\frac{x}{2}) \, dx \) (using Exercise 7) = \( \int_0^\pi \sin(\frac{x}{2}) \, dx - \int_0^\pi 2\cos^2(\frac{x}{2}) \sin(\frac{x}{2}) \, dx + \int_0^\pi \cos^4(\frac{x}{2}) \sin(\frac{x}{2}) \, dx \\
= \left[-2\cos(\frac{x}{2}) + \frac{1}{4} \cos^3(\frac{x}{2}) - \frac{x}{2} \cos^3(\frac{x}{2})\right]_0^\pi = 0 - (-2 + \frac{1}{4} - \frac{\pi}{2}) = \frac{\pi}{4} \\

9. \int \cos^3 x \, dx = \int (\cos^2 x) \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx = \int \cos x \, dx - \int \sin^2 x \cos x \, dx = \sin x - \frac{1}{3} \sin^3 x + C \\

10. \int_0^{\pi/6} 3\cos^5 3x \, dx = \int_0^{\pi/6} (\cos^2 3x)^{3/2} \cos 3x \cdot 3 \, dx = \int_0^{\pi/6} (1 - \sin^2 3x)^{3/2} \cos 3x \cdot 3 \, dx = \int_0^{\pi/6} \sin^3 3x \cos 3x \cdot 3 \, dx = \frac{\sin 3x - 2 \sin^3 3x + \sin^3 3x}{3} \bigg|_0^{\pi/6} \\
= (1 - \frac{1}{3} + \frac{1}{3}) - (0) = \frac{\pi}{4} \\

11. \int \sin^3 x \cos^3 x \, dx = \int \sin^3 x (1 - \sin^2 x) \cos x \, dx = \int \sin^3 x \cos x \, dx - \int \sin^5 x \cos x \, dx \\
= \frac{1}{4} \sin^4 x - \frac{1}{5} \sin^5 x + C \\

12. \int \cos^2 2x \sin^2 2x \, dx = \frac{1}{2} \int \cos 2x \sin 2x \, dx = \frac{1}{2} \int (1 - \sin^2 2x) \cos 2x \, dx = \frac{1}{2} \int \sin^2 2x \cos 2x \, dx - \frac{1}{2} \int \cos^2 2x \, dx = \frac{1}{12} \sin^6 2x - \frac{1}{16} \sin^8 2x + C \\

13. \int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx = \frac{1}{2} \int (1 + \cos 2x) \, dx = \frac{1}{2} \int \cos 2x \, dx = \frac{1}{2} \int \cos 2x \, dx + \frac{1}{2} \int \cos 2x \, dx = \frac{1}{3} x + \frac{1}{3} \sin 2x + C \\

14. \int_0^{\pi/2} \sin^3 x \, dx = \int_0^{\pi/2} (1 - \cos 2x) \, dx = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx = \frac{1}{2} \int_0^{\pi/2} \cos 2x \, dx = \frac{1}{2} \int_0^{\pi/2} dx - \frac{1}{4} \int_0^{\pi/2} \cos 2x \, dx = \left[\frac{1}{2}x - \frac{1}{4} \sin 2x\right]_0^{\pi/2} = \left(\frac{\pi}{4} - 0\right) - (0 - 0) = \frac{\pi}{4} \\

15. \int_0^{\pi/2} \sin^2 y \, dy = \int_0^{\pi/2} (1 - \cos^2 y) \, dy = \int_0^{\pi/2} \sin y \, dy - 3 \int_0^{\pi/2} \cos^2 y \, dy + 3 \int_0^{\pi/2} \cos^4 y \, dy = \left[-\cos y + \frac{3 \cos y}{3} - \frac{3 \cos y}{3} + \frac{3 \cos y}{3}\right]_0^{\pi/2} = (0) - (-1 + \frac{\pi}{2} + \frac{1}{2}) = \frac{\pi}{3} \\

16. \int 7 \cos^7 t \, dt (using Exercise 15) = 7 \left[\int \cos^6 t \, dt - 3 \int \sin^2 t \cos t \, dt + 3 \int \sin^4 t \sin^2 t \, dt - \int \sin^4 t \sin t \, dt\right] \\
= 7 \left(\sin t - 3 \sin^3 t + 3 \sin^5 t - \sin^7 t\right) + C = 7 \sin t - 7 \sin^3 t + \frac{7}{3} \sin^5 t - \sin^7 t + C \\

17. \int_0^\pi 8 \sin^4 x \, dx = 8 \int_0^\pi \left(1 - \cos^2 x\right)^2 \, dx = 2 \int_0^\pi (1 - 2 \cos 2x + \cos^2 2x) \, dx = 2 \int_0^\pi \cos 2x \cdot 2 \, dx + 2 \int_0^\pi \frac{1 + \cos 4x}{2} \, dx = [2x - 2 \sin 2x]_0^\pi + \int_0^\pi \cos 4x \, dx = 2 \pi + [\sin 4x]_0^\pi = 2 \pi + \pi = 3 \pi \\

18. \int_0^{\pi/2} 8 \cos^4 2x \, dx = 8 \int_0^{\pi/2} \left(1 + \cos 4x\right)^2 \, dx = 2 \int (1 + 2 \cos 4x + \cos^2 4x) \, dx = 2 \int \cos 4x \, dx + 2 \int \frac{1 + \cos 8x}{2} \, dx = 3 \int \cos 4x \, dx + \int \cos 8x \, dx = 3x + \frac{1}{8} \sin 4x + \frac{1}{8} \sin 8x \, x + C \\

19. \int_0^{\pi/2} 16 \sin^2 x \cos^2 x \, dx = 16 \int \left(\frac{1 - \cos 2x}{2}\right) \left(\frac{1 + \cos 2x}{2}\right) \, dx = 4 \int (1 - \cos^2 2x) \, dx = 4 \int dx - 4 \int \left(\frac{1 + \cos 4x}{2}\right) \, dx = 2x - 2 \int \cos 4x \, dx = 2x - 2 \int \cos 4x = 2x - \frac{1}{2} \sin 4x + C = 2x - \sin 2x \cos 2x + C = 2x - 2 \sin x \cos x (2 \cos^2 x - 1) + C = 2x - 4 \sin x \cos x + 2 \sin x \cos x + C \\

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20. \[ \int_0^\pi \sin^2 y \cos^2 y \, dy = 8 \int_0^\pi \left( \frac{1 - \cos 2y}{2} \right)^2 \left( \frac{1 + \cos 2y}{2} \right) \, dy = \int_0^\pi \cos^2 y \, dy - \int_0^\pi \cos^2 2y \, dy + \int_0^\pi \cos^2 y \, dy = \left[ -\frac{1}{2} \sin 2y \right]_0^\pi - \int_0^\pi \left( \frac{1 + \cos 4y}{2} \right) \, dy + \int_0^\pi \left( 1 - \sin^2 2y \right) \cos 2y \, dy = \pi - \frac{1}{2} \int_0^\pi \cos 4y \, dy + \int_0^\pi \cos 2y \, dy = \left[ -\frac{1}{2} \sin 4y + \frac{1}{2} \sin 2y - \frac{1}{2} \cdot \sin^2 2y \right]_0^\pi = \pi - \frac{\pi}{2} = \frac{\pi}{2} \]

21. \[ \int 8 \cos^3 2\theta \sin 2\theta \, d\theta = 8 \left( \frac{1}{2} \cos^2 \theta \right) + C = -\cos^4 \theta + C \]

22. \[ \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta \, d\theta = \int_0^{\pi/2} \sin^2 \theta (1 - \sin^2 \theta) \cos 2\theta \, d\theta = \int_0^{\pi/2} \sin^2 \theta \cos 2\theta \, d\theta - \int_0^{\pi/2} \sin^4 \theta \cos 2\theta \, d\theta = \left[ -\frac{1}{2} \sin^3 \theta \cos \theta \right]_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \sin^2 \theta \, d\theta = 0 \]

23. \[ \int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} \, dx = \int_0^{2\pi} \sin \frac{x}{2} \, dx = \left[ -2 \cos \frac{x}{2} \right]_0^{2\pi} = 2 + 2 = 4 \]

24. \[ \int_0^\pi \sqrt{1 - \cos 2x} \, dx = \int_0^\pi \sqrt{2 |\sin 2x|} \, dx = \int_0^\pi \sqrt{2} \sin 2x \, dx = \left[ -\sqrt{2} \cos 2x \right]_0^\pi = \sqrt{2} + \sqrt{2} = 2 \sqrt{2} \]

25. \[ \int_0^\pi \sqrt{1 - \sin^2 t} \, dt = \int_0^\pi |\cos t| \, dt = \int_0^{\pi/2} \cos t \, dt - \int_{\pi/2}^\pi \cos t \, dt = [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^\pi = 1 - 0 + 1 = 2 \]

26. \[ \int_0^\pi \sqrt{1 - \cos^2 \theta} \, d\theta = \int_0^\pi |\sin \theta| \, d\theta = \int_0^{\pi/2} \sin \theta \, d\theta = [\cos \theta]_0^{\pi/2} = 1 + 1 = 2 \]

27. \[ \int_{\pi/3}^{\pi/2} \frac{\sin^3 x}{\sqrt{1 - \cos x}} \, dx = \int_{\pi/3}^{\pi/2} \frac{\sin^3 x}{\sqrt{1 - \cos x}} \, dx = \int_{\pi/3}^{\pi/2} \frac{\sin^3 x}{\sqrt{1 + \cos x}} \, dx = \int_{\pi/3}^{\pi/2} \frac{\sin^3 x}{\cos^3 x} \, dx \]

28. \[ \int_0^{\pi/6} \sqrt{1 + \sin x} \, dx = \int_0^{\pi/6} \sqrt{\frac{1 + \sin x}{1 - \sin x}} \, dx = \int_0^{\pi/6} \sqrt{\frac{1 - \sin^2 x}{1 - \sin x}} \, dx = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1 - \sin x}} \, dx = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1 - \sin x}} \, dx \]

29. \[ \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sqrt{1 - \sin x}} \, dx = \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sqrt{1 - \sin x}} \, dx = \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sqrt{1 - \sin x}} \, dx = \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sqrt{1 - \sin x}} \, dx \]

30. \[ \int_{\pi/2}^{7\pi/12} \sqrt{1 - \sin x} \, dx = \int_{\pi/2}^{7\pi/12} \sqrt{1 - \sin x} \, dx = \int_{\pi/2}^{7\pi/12} \sqrt{1 - \sin x} \, dx = \int_{\pi/2}^{7\pi/12} \sqrt{\frac{1 + \sin x}{2}} \, dx = \int_{\pi/2}^{7\pi/12} \sqrt{\frac{1 + \sin x}{2}} \, dx \]
31. \[ \int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta} \, d\theta = \int_0^{\pi/2} \theta \sqrt{2} \sin \theta \, d\theta = \sqrt{2} \int_0^{\pi/2} \theta \sin \theta \, d\theta = \sqrt{2} \left[ -\theta \cos \theta + \sin \theta \right]_0^{\pi/2} = 2(1) = 2 \]

32. \[ \int_0^{\pi} (1 - \cos^2 t)^{3/2} \, dt = \int_0^{\pi} (\sin^2 t)^{3/2} \, dt = \int_0^{\pi} \sin^3 t \, dt = -\int_0^{\pi} \sin^3 t \, dt + \int_0^{\pi} \cos^2 t \sin t \, dt + \int_0^{\pi} \sin t \, dt = \left[ \cos t - \cos^3 t \right]_0^{\pi} + \left[ - \cos t + \cos^3 t \right]_0^{\pi} = (1 - \frac{1}{2} + 1 - \frac{1}{2}) + (1 - \frac{1}{2} + 1 - \frac{1}{2}) = \frac{8}{3} \]

33. \[ \int \sec^2 x \tan x \, dx = \int \tan x \sec^2 x \, dx = \frac{1}{2} \tan^2 x + C \]

34. \[ \int \sec x \tan^2 x \, dx = \int \sec x \tan x \tan x \, dx; u = \tan x, \ du = \sec^2 x \, dx, \ dv = \sec x \tan x \, dx, \ v = \sec x; \]

\[ = \sec x \tan x - \int \sec^3 x \, dx = \sec x \tan x - \int \sec^2 x \sec x \tan x \, dx = \sec x \tan x - \int (\tan^2 x + 1) \sec x \tan x \, dx \]

35. \[ \int \sec^3 x \tan x \, dx = \int \sec^2 x \sec x \tan x \, dx = \frac{1}{4} \sec^3 x + C \]

36. \[ \int \sec^4 x \tan^2 x \, dx = \int \sec^2 x (\sec^2 x - 1) \sec x \tan x \, dx = \int \sec^4 x \sec x \tan x \, dx - \int \sec^2 x \sec x \tan x \, dx = \frac{1}{2} \sec^5 x - \frac{1}{2} \sec^3 x + C \]

37. \[ \int \sec^2 x \tan^2 x \, dx = \int \tan^2 x \sec^2 x \, dx = \frac{1}{4} \tan^3 x + \frac{1}{4} \tan x + C \]

38. \[ \int \sec^3 x \tan^2 x \, dx = \int \sec^2 x \tan x \sec^2 x \, dx = \int \sec^2 x \sec x \tan x \, dx + \int \tan x \sec^2 x \, dx = \frac{1}{2} \sec^5 x + \frac{1}{2} \sec x + C \]

39. \[ \int_0^{\pi/3} 2 \sec^3 x \, dx; u = \sec x, \ du = \sec x \tan x \, dx, \ dv = \sec^2 x \, dx, \ v = \tan x; \]

\[ = 2 \sec^3 x |_0^{\pi/3} - 2 \int_0^{\pi/3} \sec x \tan^2 x \, dx = 2 \cdot 1 \cdot 0 - 2 \cdot 2 \cdot \sqrt{3} - 2 \int_0^{\pi/3} \sec x (\sec^2 x - 1) \, dx \]

\[ = 4 \sqrt{3} - 2 \int_0^{\pi/3} \sec^3 x \, dx + 2 \int_0^{\pi/3} \sec x \, dx; 2 \int_0^{\pi/3} \sec^3 x \, dx = 4 \sqrt{3} + [2 \ln |\sec x + \tan x|]_0^{\pi/3} \]

\[ = 2 \int_0^{\pi/3} \sec^3 x \, dx = 4 \sqrt{3} + 2 \ln |1 + 0| - 2 \ln |2 - \sqrt{3}| = 4 \sqrt{3} - 2 \ln (2 - \sqrt{3}) \]

\[ \int_0^{\pi/3} 2 \sec^3 x \, dx = 2 \sqrt{3} - \ln (2 - \sqrt{3}) \]

40. \[ \int e^{x^3} \sec^3 (e^x) \, dx; u = e^{x^3}, \ du = sec(e^x) \tan(e^x) e^x \, dx, \ dv = \sec^2 (e^x) e^x \, dx, \ v = \tan (e^x). \]

\[ \int e^{x^3} \sec^3 (e^x) \, dx = \sec (e^x) \tan (e^x) - \int \sec (e^x) \tan (e^x) e^x \, dx \]

\[ = \sec (e^x) \tan (e^x) - \int \sec (e^x) (\sec^2 (e^x) - 1) e^x \, dx \]

\[ = \sec (e^x) \tan (e^x) - \int \sec^3 (e^x) e^x \, dx + \int \sec (e^x) e^x \, dx \]

\[ = \sec (e^x) \tan (e^x) + \ln |\sec (e^x) + \tan (e^x)| + C \]

\[ \int e^{x^3} \sec^3 (e^x) \, dx = \frac{1}{2} (\sec (e^x) \tan (e^x) + \ln |\sec (e^x) + \tan (e^x)|) + C \]
41. \( \int \sec^4 \theta \, d\theta = \int (1 + \tan^2 \theta) \sec^2 \theta \, d\theta = \int \sec^2 \theta \, d\theta + \int \tan^2 \theta \sec^2 \theta \, d\theta = \tan \theta + \frac{1}{2} \tan^3 \theta + C \)

\(\tan \theta + \frac{1}{2} \tan \theta (\sec^2 \theta - 1) + C = \frac{1}{2} \tan \theta \sec^2 \theta + \frac{2}{3} \tan \theta + C\)

42. \(\int 3 \sec^4(3x) \, dx = \int (1 + \tan^2(3x)) \sec^2(3x) \, dx = \int \sec^2(3x) \, dx + \int \tan^2(3x) \sec^2(3x) \, dx = \tan(3x) + \frac{1}{4} \tan^3(3x) + C\)

43. \(\int_{\pi/4}^{\pi/2} \csc^4 \theta \, d\theta = \int_{\pi/4}^{\pi/2} (1 + \cot^2 \theta) \csc^2 \theta \, d\theta = \int_{\pi/4}^{\pi/2} \csc^2 \theta \, d\theta + \int_{\pi/4}^{\pi/2} \cot^2 \theta \csc^2 \theta \, d\theta = \left[-\cot \theta - \frac{\cot^3 \theta}{3}\right]_{\pi/4}^{\pi/2} = (0) - (-1 - \frac{1}{3}) = \frac{2}{3}\)

44. \(\int \sec^6 x \, dx = \int \sec^4 x \sec^2 x \, dx = \int (\tan^2 x + 1)^2 \sec^2 x \, dx = \int \left(\int (\tan^4 x + 2\tan^2 x + 1) \sec^2 x \, dx \right) = \tan x \sec x + 2 \int \sec^2 x \, dx + \int \sec^2 x \, dx = \frac{1}{3} \tan^3 x + \frac{2}{3} \tan x + C\)

45. \(\int 4 \tan^3 x \, dx = 4 \int (\sec^2 x - 1) \tan x \, dx = 4 \int \sec^2 x \tan x \, dx - 4 \int \tan x \, dx = 4 \tan^2 x - 4 \ln |\sec x| + C = 2 \tan^2 x - 4 \ln |\sec x| + C = 2 \tan^2 x - 2 \ln (1 + \tan^2 x) + C\)

46. \(\int_{-\pi/4}^{\pi/4} 6 \tan^4 x \, dx = 6 \int_{-\pi/4}^{\pi/4} (\sec^2 x - 1) \tan^2 x \, dx = 6 \int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x \, dx - 6 \int_{-\pi/4}^{\pi/4} \tan^2 x \, dx = 6 \int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x \, dx = \left[\frac{6 \tan x}{3}\right]_{-\pi/4}^{\pi/4} - 6 \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx + 6 \int_{-\pi/4}^{\pi/4} dx = 2(1 - (-1)) - [6 \tan x]_{-\pi/4}^{\pi/4} + [6 x]_{-\pi/4}^{\pi/4} = 4 - 6(1 - (-1)) + \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi - 8\)

47. \(\int \tan^5 x \, dx = \int \tan^3 x \tan x \, dx = \int (\sec^2 x - 1) \tan^2 x \, dx = \int \left(\int \sec^2 x \, dx - \tan^2 x \right) \tan x \, dx = \frac{1}{4} \sec^4 x - \sec^2 x + \ln |\sec x| + C = \frac{1}{4} (\tan^2 x + 1)^2 - (\tan^2 x + 1) + \ln |\sec x| + C = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \ln |\sec x| + C\)

48. \(\int \cot^2 x \, dx = \int \cot^4 x \cot^2 x \, dx = \int \cot^4 x (\csc^2 x - 1) \, dx = \int \cot^4 x \csc^2 x \, dx - \int \cot^4 x \, dx = \int \cot^2 x \csc^2 x \, dx - \int \cot^4 x \, dx - \int \cot^2 x \csc^2 x \, dx = \int \cot^2 x \csc^2 x \, dx - \int \cot^2 x \csc^2 x \, dx = \int \cot^2 x \csc^2 x \, dx - \int \cot^2 x \csc^2 x \, dx = \int \cot^2 x \csc^2 x \, dx - \int \cot^2 x \csc^2 x \, dx = \int \cot^2 x \csc^2 x \, dx = \left[-\cot^3 x + \ln |\csc x|\right]_{\pi/6}^{\pi/3} = \left[-\frac{\cot^3 x}{2} + \ln |\csc x|\right]_{\pi/6}^{\pi/3} = \left[-\frac{1}{2}\left(\frac{\pi}{3}\right) - 3 + \ln \frac{2}{\sqrt{3}} - \ln 2\right] = \frac{4}{3} - \ln 3\)

49. \(\int_{\pi/6}^{\pi/3} \cot^3 x \, dx = \int_{\pi/6}^{\pi/3} (\csc^2 x - 1) \cot x \, dx = \int_{\pi/6}^{\pi/3} \csc^2 x \cot x \, dx - \int_{\pi/6}^{\pi/3} \cot x \, dx = \left[-\frac{\cos^3 x}{2} + \ln |\csc x|\right]_{\pi/6}^{\pi/3} = \frac{1}{2} - 3 + \left(-\frac{\cos^3 x}{2} + \ln |\csc x|\right)_{\pi/6}^{\pi/3} = \frac{1}{2} - \ln \sqrt{3}\)

50. \(\int 8 \cot^4 t \, dt = 8 \int (\csc^2 t - 1) \cot^2 t \, dt = 8 \int \csc^2 t \cot^2 t \, dt - 8 \int \cot^2 t \, dt = -\frac{8}{3} \cot^3 t - 8 \int (\csc^2 t - 1) \, dt = -\frac{8}{3} \cot^3 t + 8 \cot t + 8 t + C\)

51. \(\int \sin 3x \cos 2x \, dx = \frac{1}{2} \int (\sin x + \sin 5x) \, dx = -\frac{1}{2} \cos x - \frac{1}{8} \cos 5x + C\)

52. \(\int \sin 2x \cos 3x \, dx = \frac{1}{2} \int (\sin(-x) + \sin 5x) \, dx = \frac{1}{2} \int (-\sin x + \sin 5x) \, dx = \frac{1}{2} \cos x - \frac{1}{4} \cos 5x + C\)
53. \( \int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos 0 - \cos 6x) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos 6x \, dx = \frac{1}{2} [x - \frac{1}{6}\sin 6x]_{-\pi}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi \)

54. \( \int_{0}^{\pi/2} \sin x \cos x \, dx = \frac{1}{2} \int_{0}^{\pi/2} (\sin 0 + \sin 2x) \, dx = \frac{1}{2} \int_{0}^{\pi/2} 2 \sin x \, dx = -\frac{1}{2} \left[ \cos 2x \right]_{0}^{\pi/2} = -\frac{1}{2} (-1 - 1) = \frac{1}{2} \)

55. \( \int \cos 3x \cos 4x \, dx = \frac{1}{2} \int (\cos (x) + \cos 7x) \, dx = \frac{1}{2} \int \cos (x) + \cos 7x \, dx = \frac{1}{2} \sin x + \frac{1}{7} \sin 7x + C \)

56. \( \int_{-\pi/2}^{\pi/2} \cos 7x \cos 8x \, dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos (6x + 8x) \, dx = \frac{1}{2} \left[ \frac{1}{6} \sin 6x + \frac{1}{8} \sin 8x \right]_{-\pi/2}^{\pi/2} = 0 \)

57. \( \int \sin^2 \theta \cos 3\theta \, d\theta = \int \frac{1 - \cos 2\theta}{2} \cos 3\theta \, d\theta = \frac{1}{2} \int \cos 3\theta \, d\theta - \frac{1}{2} \int \cos 2\theta \cos 3\theta \, d\theta \\
= \frac{1}{2} \int \cos 3\theta \, d\theta - \frac{1}{2} \int \frac{1}{2}(\cos (2 - 3)\theta + \cos (2 + 3)\theta) \, d\theta = \frac{1}{2} \int \cos 3\theta \, d\theta - \frac{1}{2} \int \cos (-\theta) + \cos 5\theta \, d\theta \\
= \frac{1}{2} \int \cos 3\theta \, d\theta - \frac{1}{2} \int \cos \theta \, d\theta - \frac{1}{4} \int \cos 5\theta + \frac{1}{2} \int \sin 3\theta - \frac{1}{2} \int \sin \theta - \frac{1}{4} \int \sin 5\theta + C \)

58. \( \int \sin^2 2\theta \sin \theta \, d\theta = \int (\cos^2 2\theta - 1) \sin \theta \, d\theta = \int (4\cos^2 2\theta - 4\cos^2 \theta + 1) \sin \theta \, d\theta \\
= \int 4\cos^2 2\theta \sin \theta \, d\theta - \int 4\cos^2 \theta \sin \theta \, d\theta + \int \sin \theta \, d\theta = -\frac{4}{3} \sin^3 \theta + \frac{4}{3} \cos^3 \theta - \cos \theta + C \)

59. \( \int \cos^3 \theta \sin 2\theta \, d\theta = \int \cos^3 \theta (2\sin \theta \cos \theta) \, d\theta = 2 \int \cos^4 \theta \sin \theta \, d\theta = -\frac{2}{3} \sin^5 \theta + C \)

60. \( \int \sin^3 \theta \cos 2\theta \, d\theta = \int \sin^2 \theta \cos 2\theta \sin \theta \, d\theta = \int \left( 1 - \cos^2 \theta \right)(2\cos^2 \theta - 1) \sin \theta \, d\theta \\
= \int (-2\cos^4 \theta + 3\cos^2 \theta - 1) \sin \theta \, d\theta = -2 \int \cos^4 \theta \sin \theta \, d\theta + 3 \int \cos^2 \theta \sin \theta \, d\theta - \int \sin \theta \, d\theta \\
= -\frac{3}{2} \cos^5 \theta - \cos^3 \theta + \cos \theta + C \)

61. \( \int \sin \theta \cos \theta \cos 3\theta \, d\theta = \frac{1}{2} \int 2\sin \theta \cos \theta \cos 3\theta \, d\theta = \frac{1}{2} \int 2\sin 2\theta \cos 3\theta \, d\theta = \frac{1}{2} \int \frac{1}{2}(\sin (2 - 3)\theta + \sin (2 + 3)\theta) \, d\theta \\
= \frac{1}{4} \int \sin (\theta) + \sin 5\theta \, d\theta = \frac{1}{4} \int (-\sin \theta + \sin 5\theta) \, d\theta = \frac{1}{4} \cos \theta - \frac{1}{20} \cos 5\theta + C \)

62. \( \int \sin \theta \sin 2\theta \sin 3\theta \, d\theta = \frac{1}{2} \int \sin (1 - 2\theta) - \cos (1 + 2\theta) \sin 3\theta \, d\theta = \frac{1}{2} \int \left( \cos (-\theta) - \cos 3\theta \right) \sin 3\theta \, d\theta \\
= \frac{1}{2} \int \sin \theta \cos 3\theta \, d\theta - \frac{1}{2} \int \sin 3\theta \cos 3\theta \, d\theta = \frac{1}{2} \int \frac{1}{2} \sin (3 - 1)\theta + \sin (3 + 1)\theta \, d\theta - \frac{1}{2} \int 2\sin 3\theta \cos 3\theta \, d\theta \\
= \frac{1}{4} \int (\sin 2\theta + \sin 4\theta) \, d\theta - \frac{1}{4} \int \sin 6\theta \, d\theta + \frac{1}{4} \int (\sin 2\theta + \sin 4\theta) \, d\theta - \frac{1}{4} \int \sin 6\theta \, d\theta \\
= -\frac{1}{8} \cos 2\theta - \frac{1}{10} \cos 4\theta + \frac{1}{7} \cos 6\theta + C \)

63. \( \int \frac{\sec^2 x}{\tan x} \, dx = \int \frac{\sec^2 x \cdot \sec x}{\tan x} \, dx = \int \frac{\tan^2 x + 1}{\tan x} \, dx = \int \frac{\tan^2 x}{\tan x} \, dx + \int \sec x \, dx = \int \tan x \sec x \, dx + \int \csc x \, dx \\
= \sec x - \ln |\csc x + \cot x| + C \)

64. \( \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{\sin^2 x \cdot \sin x}{\cos^2 x} \, dx = \int \frac{(1 - \cos^2 x) \sin x}{\cos^2 x} \, dx = \int \frac{\sin x}{\cos^2 x} \, dx - \int \frac{\cos^2 x \sin x}{\cos^2 x} \, dx = \int \sec^3 x \tan x \, dx - \int \sec x \tan x \, dx \\
= \int \sec^2 x \sec x \tan x \, dx - \int \sec x \tan x \, dx = \frac{1}{3} \sec^3 x - \sec x + C \)

65. \( \int \frac{\tan^2 x}{\cos x} \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{(1 - \cos^2 x) \sin x}{\cos^2 x} \, dx = \int \frac{1}{\cos^2 x} \sin x \, dx - \int \frac{\cos^2 x \sin x}{\cos^2 x} \, dx = \int \sec x \tan x \, dx - \int \sin x \, dx \\
= \sec x + \cos x + C \)
66. \[ \int \frac{\cot x}{\cos^3 x} \, dx = \int \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} \, dx = \int \frac{2}{2 \sin x \cos x} \, dx = \int \frac{2}{\sin^2 x} \, dx = \int \csc 2x \, 2 \, dx = -\ln|\csc 2x + \cot 2x| + C \]

67. \[ \int x \sin^7 x \, dx = \int x \frac{1 - \cos 2x}{2} \, dx = \frac{1}{2} \int x \, dx - \frac{1}{2} \int x \cos 2x \, dx \quad [u = x, \, du = dx, \, dv = \cos 2x \, dx, \, v = \frac{1}{2} \sin 2x] \]
\[ = \frac{x^2}{4} - \frac{1}{2} \left[ \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, dx \right] = \frac{x^2}{4} - \frac{1}{4} x \sin 2x - \frac{1}{4} \cos 2x + C \]

68. \[ \int x \cos^3 x \, dx = \int x \cos^2 x \cos x \, dx = \int x(1 - \sin^2 x) \cos x \, dx = \int x \cos x \, dx - \int x \sin^2 x \cos x \, dx; \]
\[ \int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + C; \]
\[ \int x \sin^2 x \cos x \, dx = \frac{1}{4} x \sin^3 x - \int \frac{1}{4} \sin^3 x \, dx; \]
\[ [u = x, \, du = dx, \, dv = \sin^3 x \cos x \, dx, \, v = \frac{1}{4} \sin^3 x] \]
\[ = \frac{x}{4} \sin^3 x - \frac{1}{4} \int (1 - \cos^2 x) \sin x \, dx = \frac{1}{4} x \sin^3 x - \frac{1}{4} \int \sin x \, dx + \frac{1}{4} \int \cos^2 x \sin x \, dx = \frac{1}{4} x \sin^3 x + \frac{1}{4} \cos x - \frac{1}{4} \cos^3 x; \]
\[ \Rightarrow \int x \cos x \, dx - \int x \sin^2 x \cos x \, dx = \left( x \sin x + \cos x \right) - \left( \frac{1}{4} x \sin^3 x + \frac{1}{4} \cos x - \frac{1}{4} \cos^3 x \right) + C \]
\[ = x \sin x - \frac{1}{4} x \sin^3 x + \frac{1}{3} \cos x + \frac{1}{4} \cos^3 x + C \]

69. \[ y = \ln(\sec x); \quad y' = \sec x \tan x = \tan^2 x; \quad \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx = \int_0^{\pi/4} |\sec x| \, dx = [\ln|\sec x + \tan x|]_0^{\pi/4} \]
\[ = \ln\left( \sqrt{2} + 1 \right) - \ln(0 + 1) = \ln\left( \sqrt{2} + 1 \right) \]

70. \[ M = \int_{-\pi/4}^{\pi/4} \sec x \, dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln\left( \sqrt{2} + 1 \right) - \ln|\sqrt{2} - 1| = \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \]
\[ y = \frac{1}{\ln \sqrt{2 + 1}} \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx = \frac{1}{\ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}} \left[ \tan x \right]_{-\pi/4}^{\pi/4} = \frac{1}{\ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}} \left( 1 - (-1) \right) = \frac{1}{\ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}} \]
\[ \Rightarrow (x, y) = \left( 0, \left( \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)^{-1} \right) \]

71. \[ V = \pi \int_0^\pi \sin^2 x \, dx = \pi \int_0^\pi \frac{1 - \cos 2x}{2} \, dx = \frac{\pi}{2} \int_0^\pi dx - \frac{\pi}{2} \int_0^\pi \cos 2x \, dx = \frac{\pi}{2} [\pi]_0^\pi - \frac{\pi}{2} [\sin 2x]_0^\pi = \frac{\pi}{2} (\pi - 0) - \frac{\pi}{2} (0 - 0) = \frac{\pi^2}{2} \]

72. \[ A = \int_0^\pi \sqrt{1 + \cos 4x} \, dx = \int_0^\pi \sqrt{2} |\cos 2x| \, dx = \sqrt{2} \int_0^\pi \cos 2x \, dx - \sqrt{2} \int_{\pi/4}^{3\pi/4} \cos 2x \, dx + \sqrt{2} \int_{3\pi/4}^\pi \cos 2x \, dx \]
\[ = \sqrt{2} [\sin 2x]_0^\pi - \sqrt{2} [\sin 2x]_{\pi/4}^{3\pi/4} + \sqrt{2} [\sin 2x]_{3\pi/4}^\pi = \sqrt{2} (1 - 0) - \sqrt{2} (-1 - 1) + \sqrt{2} (0 + 1) = \sqrt{2} + \sqrt{2} = 2 \sqrt{2} \]

73. \[ M = \int_0^{2\pi} (x + \cos x) \, dx = \left[ \frac{1}{2} x^2 + \sin x \right]_0^{2\pi} = \left( \frac{1}{2} (2\pi)^2 + \sin (2\pi) \right) - \left( \frac{1}{2} (0)^2 + \sin (0) \right) = 2\pi^2; \]
\[ x = \frac{1}{2\pi} \int_0^{2\pi} x(x + \cos x) \, dx = \frac{1}{2\pi} \int_0^{2\pi} (x^2 + x \cos x) \, dx = \frac{1}{2\pi} \int_0^{2\pi} x^2 \, dx + \frac{1}{2\pi} \int_0^{2\pi} x \cos x \, dx \]
\[ u = x, \ du = dx, \ dv = \cos x \, dx, \ v = \sin x \]
\[ = \frac{1}{2\pi} \left[ x^2 \right]_0^{2\pi} + \frac{1}{2\pi} \left[ \left( x \sin x \right) _0^{2\pi} - \int_0^{2\pi} \sin x \, dx \right] = \frac{1}{6\pi} (8\pi^3 - 0) + \frac{1}{2\pi} \left( 2\pi \sin 2\pi - 0 - \int_0^{2\pi} \sin x \, dx \right) \]
\[ = \frac{4\pi}{3} + \frac{1}{2\pi} \left[ \cos x \right]_0^{2\pi} = \frac{4\pi}{3} + \frac{1}{2\pi} (\cos 2\pi - \cos 0) = \frac{4\pi}{3} + 0 = \frac{4\pi}{3}; \quad \gamma = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} (x + \cos x)^2 \, dx \]
\[ = \frac{1}{4\pi} \int_0^{2\pi} (x^2 + 2x \cos x + \cos^2 x) \, dx = \frac{1}{4\pi} \int_0^{2\pi} x^2 \, dx + \frac{1}{2\pi} \int_0^{2\pi} x \cos x \, dx + \frac{1}{4\pi} \int_0^{2\pi} \cos^2 x \, dx \]
8.3 TRIGONOMETRIC SUBSTITUTIONS

1. $x = 3 \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = 3 \sec^2 \theta \, d\theta$, $9 + x^2 = 9(1 + \tan^2 \theta) = 9\sec^2 \theta \Rightarrow \frac{1}{\sqrt{9 + x^2}} = \frac{1}{3\sec \theta} = \frac{\cos \theta}{3} = \frac{\cos \theta}{3}$; (because $\cos \theta > 0$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$);

$$
\int \frac{dx}{\sqrt{9 + x^2}} = 3 \int \frac{\frac{d\theta}{\sec \theta}}{\sqrt{9 + \tan^2 \theta}} = \int \frac{\frac{d\theta}{\sec \theta}}{\sqrt{9 + \tan^2 \theta}} = \ln |\sec \theta + \tan \theta| + C' = \ln \left| \frac{\sqrt{9 + x^2}}{3} + \frac{3}{2} \right| + C' = \ln \left| \sqrt{9 + x^2} + x \right| + C
$$

2. $\int \frac{3 \, dx}{\sqrt{1 + 9x^2}} = \int \frac{3 \, du}{\sqrt{1 + u^2}}; u = \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, du = \frac{dt}{\sec^2 \theta}, \sqrt{1 + u^2} = |\sec \theta| = \sec \theta$;

$$
\int \frac{du}{\sqrt{1 + u^2}} = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \sqrt{1 + u^2} + u \right| + C = \ln \left| \sqrt{1 + 9x^2} + 3x \right| + C
$$

3. $\int \frac{2 \, dx}{\sqrt{9 + x^2}} = \left[ \tan^{-1} \frac{x}{3} \right]_{-2}^{2} = \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} (-1) = \left( \frac{1}{2} \right) \left( \frac{\pi}{4} \right) - \left( \frac{1}{2} \right) \left( -\frac{\pi}{4} \right) = \frac{\pi}{4}
$$

4. $\int \frac{dx}{8 + 2x^2} = \frac{1}{2} \int \frac{2 \, dx}{4 + x^2} = \frac{1}{2} \tan^{-1} \frac{x}{2} \bigg|_{0}^{2} = \frac{1}{2} \left( \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} 0 \right) = \left( \frac{1}{2} \right) \left( \frac{\pi}{4} \right) - 0 = \frac{\pi}{16}
$$

5. $\int \frac{3/2 \, dx}{\sqrt{9 - x^2}} = \left[ \sin^{-1} \frac{x}{3} \right]_{0}^{3/2} = \sin^{-1} \frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}
$$

6. $\int \frac{2 \, dx}{\sqrt{1 - 4x^2}}; \left| t = 2x \right| \rightarrow \int \frac{1/1/2 \, dt}{\sqrt{1 - t^2}} = \left[ \sin^{-1} t \right]_{0}^{1/1/2} = \sin^{-1} \frac{1}{1/2} - \sin^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}
$$

7. $t = 5 \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dt = 5 \cos \theta \, d\theta, \sqrt{25 - t^2} = 5 \cos \theta$;

$$
\int \sqrt{25 - t^2} \, dt = \int (5 \cos \theta)(5 \cos \theta) \, d\theta = 25 \int \cos^2 \theta \, d\theta = 25 \int \frac{1 + \cos 2\theta}{2} \, d\theta = 25 \left( \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) + C
$$

8. $t = \frac{1}{3} \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dt = \frac{1}{3} \cos \theta \, d\theta, \sqrt{1 - 9t^2} = \cos \theta$;

$$
\int \sqrt{1 - 9t^2} \, dt = \frac{1}{3} \int (\cos \theta)(\cos \theta) \, d\theta = \frac{1}{3} \int \cos^2 \theta \, d\theta = \frac{1}{3} \left( \theta + \sin \theta \cos \theta \right) + C = \frac{1}{3} \left[ \sin^{-1} (3t) + 3t\sqrt{1 - 9t^2} \right] + C
$$

9. $x = 7 \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = 7 \sec \theta \tan \theta \, d\theta, \sqrt{49x^2 - 49} = \sqrt{49 \sec^2 \theta - 49} = 7 \tan \theta$;

$$
\int \frac{dx}{\sqrt{4x^2 - 49}} = \int \frac{7 \sec \theta \tan \theta \, d\theta}{7 \tan \theta} = \frac{1}{2} \int \sec \theta \, d\theta = \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = \frac{1}{2} \ln \left| \frac{2x}{7} + \sqrt{4x^2 - 49} \right| + C
$$
10. \( x = \frac{3}{5} \sec \theta, 0 < \theta < \frac{\pi}{2} \), \( dx = \frac{3}{5} \sec \theta \tan \theta \, d\theta \), \( \sqrt{25x^2 - 9} = \sqrt{9 \sec^2 \theta - 9} = 3 \tan \theta \);
\[
\int \frac{5 \, dx}{\sqrt{25x^2 - 9}} = \int \left( \frac{3}{5} \sec \theta \tan \theta \bigg| \theta \bigg) \, d\theta = \int \sec \theta \, d\theta = \ln \left| \sec \theta + \tan \theta \right| + C = \ln \left| \frac{3x}{\sqrt{9}} + \frac{\sqrt{25x^2 - 9}}{3} \right| + C
\]

11. \( y = 7 \sec \theta, 0 < \theta < \frac{\pi}{2} \), \( dy = 7 \sec \theta \tan \theta \, d\theta \), \( \sqrt{y^2 - 49} = 7 \tan \theta \);
\[
\int \frac{\sqrt{y^2 - 49}}{y} \, dy = \int \left( \tan \theta \bigg| \theta \bigg) \, d\theta = 7 \int \tan^2 \theta \, d\theta = 7 \int (\sec^2 \theta - 1) \, d\theta = 7(\tan \theta - \theta) + C = 7 \left[ \frac{\sqrt{y^2 - 49}}{y} - \sec^{-1} \left( \frac{y}{7} \right) \right] + C
\]

12. \( y = 5 \sec \theta, 0 < \theta < \frac{\pi}{2} \), \( dy = 5 \sec \theta \tan \theta \, d\theta \), \( \sqrt{y^2 - 25} = 5 \tan \theta \);
\[
\int \frac{\sqrt{y^2 - 25}}{y} \, dy = \int \left( \tan \theta \bigg| \theta \bigg) \, d\theta = 5 \int \tan^2 \theta \, d\theta = 5 \int \sec^2 \theta \, d\theta = \frac{5}{10} \int (1 - \cos 2\theta) \, d\theta = \frac{1}{10} \left( \theta - \sin \theta \cos \theta \right) + C = \frac{1}{10} \left[ \sec^{-1} \left( \frac{y}{5} \right) - \left( \frac{\sqrt{y^2 - 25}}{5} \right) \right] + C
\]

13. \( x = \sec \theta, 0 < \theta < \frac{\pi}{2} \), \( dx = \sec \theta \tan \theta \, d\theta \), \( \sqrt{x^2 - 1} = \tan \theta \);
\[
\int \frac{dx}{x^2 \sqrt{x^2 - 1}} = \int \frac{\sec \theta \tan \theta \, d\theta}{\sec \theta \tan \theta} = \int \sec \theta \, d\theta = \ln \left| \sec \theta + \tan \theta \right| + C = \ln \left| \sqrt{x^2 - 1} + \frac{1}{x} \right| + C
\]

14. \( x = \sec \theta, 0 < \theta < \frac{\pi}{2} \), \( dx = \sec \theta \tan \theta \, d\theta \), \( \sqrt{x^2 - 1} = \tan \theta \);
\[
\int \frac{2 \, dx}{x^2 \sqrt{x^2 - 1}} = \int 2 \tan^2 \theta \, d\theta = 2 \int (1 + \sec^2 \theta) \, d\theta = 2 \theta + \sin \theta \cos \theta + C = \theta + \tan \theta \cos^2 \theta + C = \sec^{-1} x + \sqrt{x^2 - 1} \left( \frac{1}{x} \right)^2 + C = \sec^{-1} x + \sqrt{x^2 - 1} + C
\]

15. \( u = 9 - x^2 \Rightarrow du = -2x \, dx \Rightarrow -\frac{1}{2} \, du = x \, dx \);
\[
\int \frac{x \, dx}{\sqrt{9 - x^2}} = -\frac{1}{2} \int \frac{1}{u} \, du = -\sqrt{u} + C = -\sqrt{9 - x^2} + C
\]

16. \( x = 2 \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2} \), \( dx = 2 \sec^2 \theta \, d\theta \), \( 4 + x^2 = 4 \sec^2 \theta \);
\[
\int \frac{x^2 \, dx}{4 + x^2} = \int \frac{(2 \tan^2 \theta)^2 \sec^2 \theta \, d\theta}{4} = \int 2 \tan^2 \theta \, d\theta = \int (\sec^2 \theta - 1) \, d\theta = 2 \int \sec^2 \theta \, d\theta - 2 \int d\theta = 2 \tan \theta - 2\theta + C = x - 2 \tan^{-1} \left( \frac{x}{2} \right) + C
\]

17. \( x = 2 \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2} \), \( dx = \frac{2 \, d\theta}{\cos^2 \theta} \), \( \sqrt{x^2 + 4} = \frac{\sqrt{2}}{\cos \theta} \);
\[
\int \frac{x^2 \, dx}{\sqrt{x^2 + 4}} = \int \frac{(8 \tan^2 \theta)(\cos \theta \, d\theta)}{\cos^2 \theta} = 8 \int \tan^2 \theta \, d\theta = 8 \int (\sec^2 \theta - 1) \, d\theta = 8 \left( \sec \theta + \frac{\sec^3 \theta}{3} \right) + C = 8 \left( -\frac{\sqrt{x^2 + 4}}{2} + \frac{(x^2 + 4)^{3/2}}{8} \right) + C = \frac{1}{2} \left( x^2 + 4 \right)^{3/2} - 4 \sqrt{x^2 + 4} + C = \frac{1}{2} (x^2 - 8) \sqrt{x^2 + 4} + C
\]

18. \( x = \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2} \), \( dx = \sec^2 \theta \, d\theta \), \( \sqrt{x^2 + 1} = \sec \theta \);
\[
\int \frac{dx}{x^2 \sqrt{x^2 + 1}} = \int \frac{\sec^2 \theta \, d\theta}{\tan^2 \theta \, \sec \theta} = \int \sec \theta \, d\theta = -\frac{1}{\sin \theta} + C = -\frac{\sqrt{x^2 + 1}}{x} + C
\]

19. \( w = 2 \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2} \), \( dw = 2 \cos \theta \, d\theta \), \( \sqrt{4 - w^2} = 2 \cos \theta \);
\[
\int \frac{8 \, dw}{w^2 \sqrt{4 - w^2}} = \int \frac{8 \cos \theta \, d\theta}{4 \sin^2 \theta \cdot 2 \cos \theta} = 2 \int \frac{d\theta}{\sin^2 \theta} = -2 \cot \theta + C = -\frac{2\sqrt{4 - w^2}}{w} + C
\]
20. \( w = 3 \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \) \( dw = 3 \cos \theta \, d\theta, \) \( \sqrt{9 - w^2} = 3 \cos \theta; \)

\[
\int \frac{\sqrt{9 - w^2}}{w} \, dw = \int \frac{3 \cos \theta \cdot 3 \cos \theta}{\sin \theta} \, d\theta = \int \cot^2 \theta \, d\theta = \int \left( 1 - \sin^2 \theta \right) \, d\theta = \int (\csc^2 \theta - 1) \, d\theta
\]

\[= -\cot \theta - \theta + C = -\frac{\sqrt{9 - w^2}}{w} - \sin^{-1} \left( \frac{w}{3} \right) + C\]

21. \( u = 5x \Rightarrow du = 5dx, \ a = 6 \)

\[
\int \frac{100}{36 + 25x^2} \, dx = 20 \int \frac{1}{\left(6^2 + (5x)^2\right)} \, dx = 20 \int \frac{1}{\alpha + \omega^2} \, d\omega = 20 \cdot \frac{1}{6} \tan^{-1} \left( \frac{5x}{6} \right) + C = \frac{10}{3} \tan^{-1} \left( \frac{5x}{6} \right) + C
\]

22. \( u = x^2 - 4 \Rightarrow du = 2x \, dx \Rightarrow \frac{1}{2} \, du = x \, dx \)

\[
\int x \sqrt{x^2 - 4} \, dx = \frac{1}{2} \int \sqrt{u} \, du = \frac{1}{3} u^{3/2} + C = \frac{1}{3} (x^2 - 4)^{3/2} + C
\]

23. \( x = \sin \theta, 0 \leq \theta \leq \frac{\pi}{2}, \) \( dx = \cos \theta \, d\theta, \) \( (1 - x^2)^{3/2} = \cos^3 \theta; \)

\[
\int_0^{\sqrt{3}/2} \frac{4\sin^2 \theta \cos \theta \, d\theta}{(1 - x^2)^{3/2}} = \frac{4}{3} \left( \frac{1 - \cos^2 \theta}{\cos \theta} \right) \, d\theta = \frac{4}{3} \left( \frac{1}{\cos \theta} - \cos \theta \right) \, d\theta = \frac{4}{3} \left( \frac{1}{\sqrt{3}} - \frac{1}{3} \right)
\]

24. \( x = 2 \sin \theta, 0 \leq \theta \leq \frac{\pi}{6}, \) \( dx = 2 \cos \theta \, d\theta, \) \( (4 - x^2)^{3/2} = 8 \cos^3 \theta; \)

\[
\int_0^1 \frac{dx}{(4 - x^2)^{3/2}} = \int_0^{\pi/6} \frac{2 \cos \theta \, d\theta}{8 \cos^3 \theta} = \frac{1}{4} \int_0^{\pi/6} \frac{\cos \theta \, d\theta}{\cos^3 \theta} = \left[ \frac{1}{3} \tan \theta \right]_{0}^{\pi/6} = \frac{\sqrt{3} - 1}{4\sqrt{3}}
\]

25. \( x = \sec \theta, 0 < \theta < \frac{\pi}{2}, \) \( dx = \sec \theta \tan \theta \, d\theta, \) \( (x^2 - 1)^{3/2} = \tan^3 \theta; \)

\[
\int \frac{dx}{(x^2 - 1)^{3/2}} = \int \frac{\sec \theta \tan \theta \, d\theta}{\tan \theta} = \int \frac{\cos \theta \, d\theta}{\sin \theta} = -\frac{1}{3} \sin \theta + C = -\frac{x}{\sqrt{x^2 - 1}} + C
\]

26. \( x = \sec \theta, 0 < \theta < \frac{\pi}{2}, \) \( dx = \sec \theta \tan \theta \, d\theta, \) \( (x^2 - 1)^{5/2} = \tan^5 \theta; \)

\[
\int \frac{x \, dx}{(x^2 - 1)^{5/2}} = \int \frac{\sec^2 \theta \cdot \sec \theta \, d\theta}{\tan \theta} = \int \frac{\cos \theta \, d\theta}{\sin^2 \theta} = -\frac{1}{3} \sin \theta + C = -\frac{x^3}{3(x^2 - 1)^{3/2}} + C
\]

27. \( x = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \) \( dx = \cos \theta \, d\theta, \) \( (1 - x^2)^{3/2} = \cos^3 \theta; \)

\[
\int \frac{(1 - x^2)^{3/2} \, dx}{x} = \int \frac{\cos^3 \theta \cdot \cos \theta \, d\theta}{\cos \theta} = \int \cot^2 \theta \csc^2 \theta \, d\theta = -\frac{\cot \theta}{3} + C = -\frac{1}{3} \left( \sqrt{1 - x^2} \right)^5 + C
\]

28. \( x = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \) \( dx = \cos \theta \, d\theta, \) \( (1 - x^2)^{1/2} = \cos \theta; \)

\[
\int \frac{(1 - x^2)^{1/2} \, dx}{x} = \int \frac{\cos \theta \cos \theta \, d\theta}{\cos \theta} = \int \cot^2 \theta \csc^2 \theta \, d\theta = -\frac{\cot \theta}{3} + C = -\frac{1}{3} \left( \sqrt{1 - x^2} \right)^3 + C
\]

29. \( x = \frac{1}{2} \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \) \( dx = \frac{1}{2} \sec^2 \theta \, d\theta, \) \( (4x^2 + 1)^2 = \sec^4 \theta; \)

\[
\int \frac{8 \, dx}{(4x^2 + 1)^{3/2}} = \int \frac{8 \left( \frac{1}{2} \sec^2 \theta \right) \, d\theta}{\sec^4 \theta} = 4 \int \cos^2 \theta \, d\theta = 2(\theta + \sin \theta \cos \theta) + C = 2 \tan^{-1} 2x + \frac{4x}{(4x^2 + 1)} + C
\]

30. \( t = \frac{1}{2} \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \) \( dt = \frac{1}{3} \sec^2 \theta \, d\theta, \) \( 9t^2 + 1 = \sec^2 \theta; \)

\[
\int \frac{6 \, dt}{(9t^2 + 1)^{1/2}} = \int \frac{6 \left( \frac{1}{3} \sec^2 \theta \right) \, d\theta}{\sec^4 \theta} = 2 \int \cos^2 \theta \, d\theta = \theta + \sin \theta \cos \theta + C = \tan^{-1} 3t + \frac{3t}{(9t^2 + 1)} + C
\]

31. \( u = x^2 - 1 \Rightarrow du = 2x \, dx \Rightarrow \frac{1}{2} \, du = x \, dx \)

\[
\int \frac{x^3}{x^2 - 1} \, dx = \int \left( x + \frac{x}{x^2 - 1} \right) \, dx = \int x \, dx + \int \frac{x}{x^2 - 1} \, dx = \frac{x^2}{2} + \frac{1}{2} \int \frac{4}{\left( \frac{x^2}{4} - 1 \right)} \, dx = \frac{1}{2} x^2 + \frac{1}{2} \ln |u| + C = \frac{1}{2} x^2 + \frac{1}{2} \ln |x^2 - 1| + C
\]
32. \( u = 25 + 4x^2 \Rightarrow du = 8x \, dx \Rightarrow \frac{1}{8} \, du = x \, dx \)
\[
\int \frac{x}{25 + 4x^2} \, dx = \frac{1}{8} \int \frac{1}{u} \, du = \frac{1}{8} \ln |u| + C = \frac{1}{8} \ln (25 + 4x^2) + C
\]

33. \( v = \sin \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \, dv = \cos \theta \, d\theta, \quad (1 - v^2)^{1/2} = \cos \theta; \)
\[
\int \frac{v^2 \, dv}{(1 - v^2)^{1/2}} = \int \frac{\sin^2 \theta \, \cos \theta \, d\theta}{\cos \theta} = \int \tan^2 \theta \, \sec^2 \theta \, d\theta = \tan \theta + C = \frac{1}{2} \left( \frac{x}{\sqrt{1-x^2}} \right)^3 + C
\]

34. \( r = \sin \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}; \)
\[
\int \frac{1 - r^2 \, dr}{r^2} = \int \frac{\cos^2 \theta \, \cos \theta \, d\theta}{\sin^2 \theta} = \int \cot^2 \theta \, \csc^2 \theta \, d\theta = -\cot \theta + C = -\frac{1}{2} \left[ \frac{\sqrt{1-x^2}}{x} \right] + C
\]

35. Let \( e^3 = 3 \tan \theta, \quad t = \ln (3 \tan \theta), \quad \tan \theta = t \Rightarrow \sec^2 \theta \, d\theta = \frac{1}{3} \, e^3 \, dt, \quad 1 + e^6 = 1 + \tan^2 \theta = \sec^2 \theta; \)
\[
\int_{\ln(3/4)}^{3\ln(3/4)} \frac{e^3 \, dt}{(1 + e^6)^{1/4}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(1/2)} \frac{\tan \theta \, \sec^2 \theta \, d\theta}{\tan \theta \cdot 3 \sec \theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(1/2)} \sec \theta \, d\theta = \left[ \sec \theta \right]_{\tan^{-1}(3/4)}^{\tan^{-1}(1/2)} = \frac{\sqrt{2} - \sqrt{1/2}}{2} \left( \frac{\pi}{4} - \frac{\pi}{6} \right)
\]

36. Let \( e^3 = 3 \tan \theta, \quad t = \ln (\tan \theta), \quad \tan^{-1}(1/2) \leq \theta \leq \tan^{-1}(3/4), \, dt = \sec^2 \theta \, d\theta, \quad 1 + e^6 = 1 + \tan^2 \theta = \sec^2 \theta; \)
\[
\int_{\ln(1/2)}^{\ln(3/4)} \frac{e^3 \, dt}{(1 + e^6)^{1/4}} = \int_{\tan^{-1}(1/2)}^{\tan^{-1}(3/4)} \frac{\tan \theta \, \sec^2 \theta \, d\theta}{\sec \theta} = \int_{\tan^{-1}(1/2)}^{\tan^{-1}(3/4)} \cos \theta \, d\theta = \left[ \sin \theta \right]_{\tan^{-1}(1/2)}^{\tan^{-1}(3/4)} = \frac{4}{3} \left( \frac{\pi}{3} - \frac{\pi}{4} \right)
\]

37. \( y = e^{-\tan \theta}, \quad 0 \leq \theta \leq \frac{\pi}{2}, \, dy = e^{-\tan \theta} \sec^2 \theta \, d\theta, \quad \sqrt{1 + (\ln y)^2} = \sqrt{1 + \tan^2 \theta} = \sec \theta; \)
\[
\int_{\ln(1/2)}^{\ln(3/4)} \frac{dy}{\sqrt{1 + (\ln y)^2}} = \int_{\tan^{-1}(1/2)}^{\tan^{-1}(3/4)} \frac{e^{-\tan \theta} \sec^2 \theta \, d\theta}{\sec \theta} = \int_{\tan^{-1}(1/2)}^{\tan^{-1}(3/4)} \cos \theta \, d\theta = \left[ \sec \theta \right]_{\tan^{-1}(1/2)}^{\tan^{-1}(3/4)} = \frac{2}{3} \pi - \frac{\pi}{6}
\]

38. \( x = \frac{1}{2} \ln(1 + \tan \theta), \quad 0 \leq \theta \leq \frac{\pi}{2}, \, dx = \frac{1}{2} \ln(1 + \tan \theta) \, d\theta, \quad \sqrt{1 + (\ln x)^2} = \sqrt{1 + \tan^2 \theta} = \sec \theta; \)
\[
\int_{1/2}^{1/4} \frac{dy}{\sqrt{1 + (\ln y)^2}} = \int_{\tan^{-1}(1/2)}^{\tan^{-1}(1/4)} \frac{e^{\tan \theta} \sec^2 \theta \, d\theta}{\sec \theta} = \int_{\tan^{-1}(1/2)}^{\tan^{-1}(1/4)} \sec \theta \, d\theta = \left[ \sec \theta \right]_{\tan^{-1}(1/2)}^{\tan^{-1}(1/4)} = \ln \left( 1 + \sqrt{2} \right)
\]

39. \( x = \sec \theta, \quad 0 < \theta < \frac{\pi}{2}, \, dx = \sec \theta \, \tan \theta \, d\theta, \quad \sqrt{x^2 - 1} = \sec \theta - 1 = \tan \theta; \)
\[
\int \frac{dx}{\sqrt{x^2 - 1}} = \int \frac{\sec \theta \, \tan \theta \, d\theta}{\sec \theta \, \tan \theta} = \theta + C = \sec^{-1} x + C
\]

40. \( x = \tan \theta, \quad dx = \sec^2 \theta \, d\theta, \quad 1 + x^2 = \sec^2 \theta; \)
\[
\int \frac{dx}{x^2 + 1} = \int \frac{\sec^2 \theta \, d\theta}{\sec^2 \theta} \sec \theta = \theta + C = \tan^{-1} x + C
\]

41. \( x = \sec \theta, \quad dx = \sec \theta \, \tan \theta \, d\theta, \quad \sqrt{x^2 - 1} = \sec \theta - 1 = \tan \theta; \)
\[
\int \frac{dx}{\sqrt{x^2 - 1}} = \int \frac{\sec \theta \, \tan \theta \, d\theta}{\sec \theta} = \int \sec \theta \, d\theta = \tan \theta + C = \sqrt{x^2 - 1} + C
\]

42. \( x = \sin \theta, \quad dx = \cos \theta \, d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}; \)
\[
\int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{\cos \theta \, d\theta}{\cos \theta} = \theta + C = \sin^{-1} x + C
\]

43. \( x^2 = \tan \theta, \quad 0 \leq \theta < \frac{\pi}{2}, \quad 2x \, dx = \sec^2 \theta \, d\theta \Rightarrow x \, dx = \frac{1}{2} \sec^2 \theta \, d\theta; \quad \sqrt{1 + x^4} = \sqrt{1 + \tan^2 \theta} = \sec \theta \)
\[
\int \frac{x}{\sqrt{1 + x^4}} \, dx = \frac{1}{2} \int \frac{\sec^2 \theta \, d\theta}{\sec \theta} = \frac{1}{2} \int \sec \theta \, d\theta = \frac{1}{2} \ln|\sec \theta + \tan \theta| + C = \frac{1}{2} \ln|\sqrt{1 + x^4} + x^2| + C
\]
44. Let \( \ln x = \sin \theta \), \( -\frac{\pi}{2} \leq \theta < 0 \) or \( 0 < \theta \leq \frac{\pi}{2} \). 
\[
\frac{1}{\sqrt{1-(\ln x)^2}} \frac{dx}{x} = \cos \theta \, d\theta, \quad \sqrt{1-(\ln x)^2} = \cos \theta
\]
\[
\int \sqrt{\frac{1-(\ln x)^2}{\ln x}} \frac{dx}{x} = \int \frac{\cos \theta}{\sin \theta} \, d\theta = \int \csc \theta \, d\theta - \int \sin \theta \, d\theta = -\ln|\csc \theta + \cot \theta| + \cos \theta + C
\]
\[
= -\ln \left(1 + \sqrt{1-(\ln x)^2}\right) + \sqrt{1-(\ln x)^2} + C = -\ln \left(1 + \sqrt{1-(\ln x)^2}\right) + \sqrt{1-(\ln x)^2} + C
\]

45. Let \( u = x = u^2 \Rightarrow \frac{du}{dx} = 2u \), \( du = 2u \, dx \Rightarrow \int \sqrt{\frac{1-x^2}{x^2}} \, dx = \int \frac{\sqrt{1-u^2}}{1-u^2} \, du = 2 \int \frac{\sqrt{1-u^2}}{u^2} \, du; \)
\[
u_1 = u, \quad \frac{1}{u} = \frac{1}{u} \Rightarrow \frac{du}{dx} = 2u \Rightarrow \int \sqrt{\frac{1-x^2}{x^2}} \, dx = \int \frac{\sqrt{1-u^2}}{1-u^2} \, du = 2 \int \frac{\sqrt{1-u^2}}{u^2} \, du = 2 \int \frac{\sqrt{1-u^2}}{u^2} \, du,
\]
\[
u = \sin \theta, \quad du = \cos \theta \, d\theta, \quad -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}, \quad \sqrt{1-u^2} = \cos \theta
\]
\[
2 \int \sqrt{1-u^2} \, du = 2 \int \sin^2 \theta \, d\theta = 2 \int (\cos^2 \theta) \, d\theta = 2 \int \cos^2 \theta \, d\theta = 2 \left(\frac{1}{2} \int \sin^2 \theta \, d\theta + \frac{1}{2} \int \cos^2 \theta \, d\theta\right) = \frac{1}{2} \int \sin^2 \theta \, d\theta + \frac{1}{2} \int \cos^2 \theta \, d\theta
\]
\[
= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{8}\right) = \frac{1}{8} \sin \theta \cos \theta + C = \frac{1}{2}\theta - \frac{1}{8} \frac{1}{2} \sin 2 \theta + C = \frac{1}{2}\theta - \frac{1}{8} \frac{1}{2} \sin 2 \theta + C
\]
\[
= \frac{1}{4} \sin^{-1} \sqrt{x} + \frac{1}{8} \sqrt{(1-x)^{3/2}} + \frac{1}{4} \sqrt{x} \frac{1}{2} + C
\]

46. Let \( w = x = w^2 \Rightarrow \frac{dw}{dx} = 2u \), \( du = 2u \, dw \Rightarrow \int \sqrt{\frac{1-x^2}{x^2}} \, dx = \int u \sqrt{1-u^2} \, du = 2 \left(\frac{1}{2} \int \frac{1}{1-u^2} \, du + \frac{1}{2} \int \frac{1}{1-u^2} \, du\right) = \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1} (x^{3/2}) + C
\]

47. Let \( u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow \frac{dx}{du} = 2u \Rightarrow \int \sqrt{x} \sqrt{1-x} \, dx = \int u \sqrt{1-u^2} \, du = 2 \left(\frac{1}{2} \int \frac{1}{1-u^2} \, du + \frac{1}{2} \int \frac{1}{1-u^2} \, du\right) = \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1} (x^{3/2}) + C
\]

48. Let \( w = x = w^2 \Rightarrow w^2 = x = 1 \Rightarrow 2w \, dw = dx \Rightarrow \int \sqrt{\frac{x^2-1}{x^2-1}} \, dx = \int \frac{\sqrt{w^2-1}}{w^2-1} \, dw = 2 \left(\frac{1}{2} \int \frac{1}{1-w^2} \, dw + \frac{1}{2} \int \frac{1}{1-w^2} \, dw\right) = \frac{1}{2} \sin^{-1} w + C = \frac{1}{2} \sin^{-1} (\sqrt{w^2-1}) + C
\]

49. \( \frac{dy}{dx} = \sqrt{x^2-4}; \quad dy = \sqrt{x^2-4} \, dx \); \( y = \int \sqrt{x^2-4} \, dx \);
\[
\frac{dy}{dx} = \sqrt{x^2-4}; \quad \frac{dy}{dx} = 2 \frac{2 \sec \theta \tan \theta \, d\theta}{\sec \theta} = 2 \left(\frac{2 \tan \theta}{2 \sec \theta} \right) \, d\theta = 2 \left(\frac{2 \sec \theta \tan \theta}{2 \sec \theta} \right) \, d\theta = 2 \tan \theta + C
\]
\[
= 2 \left[\sqrt{x^2-4} - \sec^{-1} \left(\frac{x}{2}\right)\right] + C; \quad x = 2 \text{ and } y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2 \left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1} \left(\frac{x}{2}\right)\right]
\]
50. \( \sqrt{x^2 - 9} \frac{dy}{dx} = 1 \), dy = \( \frac{dx}{\sqrt{x^2 - 9}} \); \( y = \int \frac{dx}{\sqrt{x^2 - 9}} \); \( x = 3 \sec \theta, 0 < \theta < \frac{\pi}{2} \)
\[ dx = 3 \sec \theta \tan \theta \, d\theta \]
\[ y = \int \frac{3 \sec \theta \tan \theta \, d\theta}{\sqrt{9} \tan \theta} \]
\[ = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{x + \sqrt{x^2 - 9}}{3} \right| + C; x = 5 \text{ and } y = \ln 3 \implies \ln 3 = \ln 3 + C \implies C = 0 \]
\[ \implies y = \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \]

51. \( (x^2 + 4) \frac{dy}{dx} = 3 \), dy = \( \frac{dx}{x^2 + 4} \); \( x = \tan \theta, dx = \sec^2 \theta \, d\theta \), \( (x^2 + 1)^{3/2} = \sec^3 \theta \);
\[ y = \int \sec^2 \theta \, d\theta = \int \cos \theta \, d\theta = \sin \theta + C = \tan \theta \cos \theta + C = \frac{\sin \theta \cos \theta}{\sqrt{x^2 + 1}} + C; x = 0 \text{ and } y = 1 \]
\[ \implies 1 = 0 + C \implies y = \frac{1}{\sqrt{x^2 + 1}} + 1 \]

52. \( (x^2 + 1)^{3/2} \frac{dy}{dx} = \sqrt{x^2 + 1} \), dy = \( \frac{dx}{x^2 + 1} \); \( x = \tan \theta, dx = \sec^2 \theta \, d\theta \), \( x^2 + 1 \); \( (x^2 + 1)^{3/2} = \sec^3 \theta \);
\[ y = \int \sec^2 \theta \, d\theta = \int \cos \theta \, d\theta = \sin \theta + C = \tan \theta \cos \theta + C = \frac{\tan \theta + \sec \theta}{x^2 + 1} + C; x = 0 \text{ and } y = 1 \]
\[ \implies 1 = 0 + C \implies y = \frac{1}{\sqrt{x^2 + 1}} + 1 \]

53. A = \( \int_0^\pi \sqrt{\frac{9 - x^2}{3}} \, dx \); \( x = 3 \sin \theta, 0 \leq \theta \leq \frac{\pi}{2} \), dx = 3 \cos \theta \, d\theta, \sqrt{9 - x^2} = \sqrt{9 - 9\sin^2 \theta} = 3 \cos \theta ;
A = \( \int_0^{\pi/2} 3 \cos \theta \cdot 3 \cos \theta \, d\theta = 3 \int_0^{\pi/2} \cos^2 \theta \, d\theta = \frac{3}{2} \left[ \theta + \sin \theta \cos \theta \right]_0^{\pi/2} = \frac{3\pi}{4} \)

54. \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies y = \pm b \sqrt{1 - \frac{x^2}{a^2}} \); A = \( 4 \int_0^a b \sqrt{1 - \frac{x^2}{a^2}} \, dx = 4b \int_0^a \sqrt{1 - \frac{x^2}{a^2}} \, dx \)
\[ x = \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad dx = \cos \theta \, d\theta, \quad 1 - \frac{x^2}{a^2} = \cos \theta, \quad x = 0 \implies \sin \theta = 0, \quad x = a \implies \sin \theta = \frac{\pi}{2} \]
\begin{align*}
4b \int_0^a \sqrt{1 - \frac{x^2}{a^2}} \, dx &= 4b \int_0^{\pi/2} \cos \theta \left( a \cos \theta \right) \, d\theta = 4ab \int_0^{\pi/2} \cos^2 \theta \, d\theta = 4ab \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} \, d\theta \\
&= 2ab \int_0^{\pi/2} d\theta + 2ab \int_0^{\pi/2} \cos 2\theta \, d\theta = 2ab \left[ \theta \right]_0^{\pi/2} + ab \left[ \sin 2\theta \right]_0^{\pi/2} = 2ab \left( \frac{\pi}{2} - 0 \right) + ab \left( \sin \pi - \sin 0 \right) = \pi ab
\end{align*}

55. (a) A = \( \int_0^{1/2} \sin^{-1} x \, dx \left[ u = \sin^{-1} x, \; du = \frac{1}{\sqrt{1 - x^2}} \, dx, \; dv = dx, \; v = x \right] \)
\[ = \left[ x \sin^{-1} x \right]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1 - x^2}} \, dx \quad = \left( \frac{1}{2} \sin^{-1} \frac{1}{2} - 0 \right) + \left[ \sqrt{1 - x^2} \right]_0^{1/2} = \frac{\pi + 6\sqrt{3} - 12}{12} \]

(b) M = \( \int_0^{1/2} \sin^{-1} x \, dx = \frac{\pi + 6\sqrt{3} - 12}{12}; \quad x = \frac{1}{\sqrt{1 - x^2}} \int_0^{1/2} x \sin^{-1} x \, dx = \left[ u = \sin^{-1} x, \; du = \frac{1}{\sqrt{1 - x^2}} \, dx, \; dv = dx, \; v = x \right] \)
\begin{align*}
\frac{12}{\pi + 6\sqrt{3} - 12} \left[ \left( \frac{1}{2} x^2 \sin^{-1} x \right)_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1 - x^2}} \, dx \right] \\
&= \frac{12}{\pi + 6\sqrt{3} - 12} \left( \frac{1}{2} \left( \frac{1}{2} \right)^2 \sin^{-1} \left( \frac{1}{2} \right) - 0 \right) - \int_0^{\pi/6} \frac{\sin \theta \, d\theta}{\cos \theta} = \frac{12}{\pi + 6\sqrt{3} - 12} \left( \frac{\pi}{4} - \frac{1}{2} \int_0^{\pi/6} \sin^2 \theta \, d\theta \right) \\
&= \frac{12}{\pi + 6\sqrt{3} - 12} \left( \frac{\pi}{4} - \frac{1}{2} \int_0^{\pi/6} \cos 2\theta \, d\theta \right) = \frac{12}{\pi + 6\sqrt{3} - 12} \left( \frac{\pi}{4} - \frac{1}{4} \int_0^{\pi/6} \cos 2\theta \, d\theta + \frac{1}{4} \int_0^{\pi/6} \cos \theta \, d\theta \right) \\
&= \frac{12}{\pi + 6\sqrt{3} - 12} \left( \frac{\pi}{4} + \frac{\theta}{4} + \frac{\sin 2\theta}{2} \right) = \frac{3\sqrt{3} - \pi}{4(\pi + 6\sqrt{3} - 12)}; \quad y = \frac{1}{\pi + 6\sqrt{3} - 12} \int_0^{1/2} \frac{1}{2} (\sin^{-1} x)^2 \, dx \\
&= \left[ (\sin^{-1} x)^2, \; du = \frac{2\sin^{-1} x}{\sqrt{1 - x^2}} \, dx, \; dv = dx, \; v = x \right] \]

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(a) The slope of the tangent line is also 
\[ \frac{\sin x}{\cos x} = \tan x, \]
thus
\[ \frac{dx}{\cos x} = \sec x \, dx. \]

(b) Substitution: 
\[ u = \tan x, \quad du = \sec^2 x \, dx, \quad dv = dx, \quad v = x \]

57. (a) Integration by parts: 
\[ u = x^2, \quad du = 2x \, dx, \quad dv = x \sqrt{1 - x^2} \, dx, \quad v = -\frac{1}{2}(1 - x^2)^{3/2} \]
\[ \int x^3 \sqrt{1 - x^2} \, dx = -\frac{1}{2}x^2(1 - x^2)^{3/2} + \frac{1}{3} \int (1 - x^2)^{3/2} 2x \, dx = -\frac{1}{6}x^3(1 - x^2)^{3/2} - \frac{2}{3} \frac{1}{2}(1 - x^2)^{5/2} + C \]
(b) Substitution: 
\[ u = 1 - x^2 \Rightarrow x^2 = 1 - u \Rightarrow du = -2x \, dx \Rightarrow -\frac{1}{2} \, dx = u \, dx \]
\[ \int x^3 \sqrt{1 - x^2} \, dx = \int x^2 \sqrt{1 - x^2} \, dx = -\frac{1}{2} \int (1 - u) \sqrt{u} \, du = -\frac{1}{2} \int (\sqrt{u} - u^{3/2}) \, du = -\frac{1}{4}u^{3/2} + \frac{1}{6}u^{5/2} + C \]
\[ = -\frac{1}{4}(1 - x^2)^{3/2} + \frac{1}{6}(1 - x^2)^{5/2} + C \]
(c) Trig substitution: 
\[ x = \sin \theta, \quad \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad dx = \cos \theta \, d\theta, \quad \sqrt{1 - x^2} = \cos \theta \]
\[ \int x^3 \sqrt{1 - x^2} \, dx = \int \sin^3 \theta \cos \theta \, d\theta = \int \sin^2 \theta \cos^2 \theta \, d\theta = \int (1 - \cos^2 \theta) \cos^2 \theta \, d\theta = \int \cos^2 \theta \, d\theta = \int \cos^4 \theta \, d\theta = \frac{1}{4} \cos^3 \theta + \frac{1}{3} \cos^3 \theta + C = \frac{1}{4}(1 - x^2)^{3/2} + \frac{7}{10}(1 - x^2)^{5/2} + C \]

58. (a) The slope of the line tangent to \( y = f(x) \) is given by \( f'(x) \). Consider the triangle whose hypotenuse is the 30 ft rope, the length of the base is \( x \) and the height \( h = \sqrt{900 - x^2} \). The slope of the tangent line is also \( -\frac{\sqrt{900 - x^2}}{x} \), thus
\[ f'(x) = -\frac{\sqrt{900 - x^2}}{x}. \]
(b) 
\[ f(x) = \int -\frac{\sqrt{900 - x^2}}{x} \, dx \quad \left[ x = 30 \sin \theta, \quad 0 < \theta \leq \frac{\pi}{2}, \quad dx = 30 \cos \theta \, d\theta, \quad \sqrt{900 - x^2} = 30 \cos \theta \right] \]
\[ = -\int 30 \cos \theta \cdot \frac{30 \cos \theta}{30 \sin \theta} \, d\theta = -30 \int \cos^2 \theta \, d\theta = -30 \int \left( \frac{1 - \sin^2 \theta}{\sin \theta} \right) \, d\theta = -30 \int \csc \theta \, d\theta + 30 \int \sin \theta \, d\theta \]
\[ = 30 \ln|\csc \theta + \cot \theta| - 30 \cos \theta + C = 30 \ln \left| \frac{30}{x} + \frac{\sqrt{900 - x^2}}{x} \right| - \sqrt{900 - x^2} + C; \quad f(30) = 0 \]
\[ \Rightarrow 0 = 30 \ln \left| \frac{30}{x} + \frac{\sqrt{900 - 30^2}}{x} \right| - \sqrt{900 - 30^2} + C \Rightarrow f(x) = 30 \ln \left| \frac{30}{x} + \frac{\sqrt{900 - x^2}}{x} \right| - \sqrt{900 - x^2} \]

8.4 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

1. 
\[ \frac{5x - 13}{(x - 3)(x - 2)} = \frac{A}{x - 3} + \frac{B}{x - 2} \quad \Rightarrow \quad 5x - 13 = A(x - 2) + B(x - 3) = (A + B)x - (2A + 3B) \]
\[ \Rightarrow \quad A + B = 5 \quad \text{and} \quad 2A + 3B = 13 \]
\[ \Rightarrow \quad -B = 10 - 13 \quad \Rightarrow \quad B = 3 \quad \Rightarrow \quad A = 2; \quad \text{thus,} \quad \frac{5x - 13}{(x - 3)(x - 2)} = \frac{2}{x - 3} + \frac{3}{x - 2} \]

2. 
\[ \frac{5x - 7}{x^2 - 3x + 2} = \frac{5x - 7}{(x - 2)(x - 1)} = \frac{A}{x - 2} + \frac{B}{x - 1} \quad \Rightarrow \quad 5x - 7 = A(x - 1) + B(x - 2) = (A + B)x - (A + 2B) \]
\[ \Rightarrow \quad A + B = 5 \quad \text{and} \quad A + 2B = 7 \]
\[ \Rightarrow \quad B = 2 \quad \Rightarrow \quad A = 3; \quad \text{thus,} \quad \frac{5x - 7}{x^2 - 3x + 2} = \frac{3}{x - 2} + \frac{2}{x - 1} \]
3. \( \frac{x^4 + 4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x + 4 = A(x + 1) + B \) \( A + B = 4 \) \( \Rightarrow A = 1 \) and \( B = 3; \) thus, \( \frac{x^4 + 4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2} \)

4. \( \frac{2x^2 + 2}{x^2 - 2x + 1} = \frac{A}{x-1} + \frac{B}{x-1} \Rightarrow 2x + 2 = A(x - 1) + B \) \( A + B = 2 \) \( \Rightarrow A = 2 \) and \( B = 4; \) thus, \( \frac{2x^2 + 2}{x^2 - 2x + 1} = \frac{2}{x-1} + \frac{4}{x-1} \)

5. \( \frac{z + 1}{z^2 - 1} = \frac{A}{z - 1} + \frac{B}{z + 1} \Rightarrow z + 1 = A(z - 1) + B(z + 1) \) \( A + C = 0 \) \( \Rightarrow -A + B = 1 \) \( \Rightarrow B = -1 \) \( \Rightarrow A = -2 \) \( \Rightarrow C = 2; \) thus, \( \frac{z + 1}{z^2 - 1} = \frac{2}{z - 1} + \frac{2}{z + 1} \)

6. \( \frac{2x + 2}{x^2 - 3x - 2} = \frac{A}{x - 1} + \frac{B}{x - 2} \Rightarrow 2x + 2 = A(x - 2) + B(x - 1) \) \( A + B = 2 \) \( \Rightarrow A = 2 \) and \( B = 4; \) thus, \( \frac{2x + 2}{x^2 - 3x - 2} = \frac{2}{x - 1} + \frac{4}{x - 2} \)

7. \( \frac{t^2 + 6}{t^2 - 3t + 2} = 1 + \frac{5t + 2}{t^2 - 3t + 2} \) (after long division); \( \frac{5t + 2}{t^2 - 3t + 2} = \frac{5t + 2}{t - 1}(t - 2) \) \( \Rightarrow 5t + 2 = A(t - 2) + B(t - 3) \) \( A + B = 5 \) \( \Rightarrow -2A - 3B = 2 \) \( \Rightarrow B = -\frac{1}{3} \) \( \Rightarrow A = \frac{17}{3}; \) thus, \( \frac{t^2 + 6}{t^2 - 3t + 2} = 1 + \frac{17}{3} + \frac{12}{t^2 - 3t + 2} \)

8. \( \frac{t^2 + 9}{t^2 + 9} = 1 + \frac{9t^2 + 9}{t^2 + 9} \) \( \Rightarrow 9t^2 + 9 = A(t + 3) + B(t - 3) \) \( A + B = 9 \) \( \Rightarrow 9A = 0 \) \( \Rightarrow 9B = 9 \) \( \Rightarrow B = 1 \) \( \Rightarrow C = 0; \) thus, \( \frac{t^2 + 9}{t^2 + 9} = 1 + \frac{9}{t^2 + 9} \)

9. \( \frac{1}{x - x^2} = \frac{A}{x} + \frac{B}{1 - x} \Rightarrow 1 = A(1 + x) + B(1 - x); x = 1 \Rightarrow A = \frac{1}{2}; x = -1 \Rightarrow B = \frac{1}{2}; \) \( \int \frac{1}{x^2 - x} \, dx = \frac{1}{2} \int \frac{1}{x} + \frac{1}{x - 1} = \frac{1}{2} \ln |1 + x| - \ln |1 - x| + C \)

10. \( \frac{1}{x^2 + 2x} = \frac{A}{x} + \frac{B}{x + 2} \Rightarrow 1 = A(x + 2) + Bx; x = 0 \Rightarrow A = \frac{1}{2}; x = -2 \Rightarrow B = -\frac{1}{2}; \) \( \int \frac{1}{x^2 + 2x} \, dx = \frac{1}{2} \int \frac{1}{x} - \frac{1}{x + 2} = \frac{1}{2} \ln |x| - \ln |x + 2| + C \)

11. \( \frac{x + 4}{x^2 + 5x - 6} = \frac{A}{x + 6} + \frac{B}{x - 1} \Rightarrow x + 4 = A(x - 1) + B(x + 6); x = 1 \Rightarrow B = \frac{5}{7}; x = -6 \Rightarrow A = -\frac{2}{7}; \) \( \int \frac{x + 4}{x^2 + 5x - 6} \, dx = \frac{2}{7} \int \frac{1}{x + 6} + \frac{5}{7} \int \frac{1}{x - 1} = \frac{2}{7} \ln |x + 6| + \frac{5}{7} \ln |x - 1| + C = \frac{2}{7} \ln |(x + 6)^2(x - 1)| + C \)

12. \( \frac{2x + 1}{x^2 - 3x - 2} = \frac{A}{x - 4} + \frac{B}{x - 3} \Rightarrow 2x + 1 = A(x - 3) + B(x - 4); x = 3 \Rightarrow B = \frac{7}{4}; x = 4 \Rightarrow A = \frac{9}{4}; \) \( \int \frac{2x + 1}{x^2 - 3x - 2} \, dx = 7 \int \frac{1}{x - 4} - 8 \int \frac{1}{x - 3} = 9 \ln |x - 4| - 7 \ln |x - 3| + C = \ln \left| \frac{x - 4}{x - 3} \right| + C \)

13. \( \frac{y^2}{y^2 - 3} = \frac{A}{y - 3} + \frac{B}{y + 1} \Rightarrow y = A(y + 1) + B(y - 3); y = -1 \Rightarrow B = \frac{1}{4}; y = 3 \Rightarrow A = \frac{1}{4}; \) \( \int \frac{y^2}{y^2 - 3} \, dy = \frac{1}{4} \int \frac{1}{y - 3} + \frac{1}{4} \int \frac{1}{y + 1} = \left[ \frac{1}{4} \ln |y - 3| + \frac{1}{4} \ln |y + 1| \right]_{4}^{5} = \left( \frac{1}{4} \ln 5 + \frac{1}{4} \ln 9 \right) - \left( \frac{1}{4} \ln 1 + \frac{1}{4} \ln 5 \right) = \frac{1}{4} \ln 5 + \frac{1}{7} \ln 3 = \ln \frac{15}{2} \)
14. \( \frac{x^4 + 4}{x^2 + 2} = \frac{A}{x^2 + 2} + \frac{B}{x + 1} \Rightarrow y + 4 = A(y + 1) + By; y = 0 \Rightarrow A = 4; y = -1 \Rightarrow B = \frac{3}{7} = -3; \)

\[
\int_{1/2}^{1} \frac{x^4 + 4}{x^2 + 2} \, dy = 4 \int_{1/2}^{1} \frac{dy}{y} - 3 \int_{1/2}^{1} \frac{dy}{y + 1} = [4 \ln|y| - 3 \ln|y + 1|]_{1/2} = (4 \ln 1 - 3 \ln 2) - (4 \ln \frac{1}{2} - 3 \ln \frac{3}{2}) = \ln \frac{3}{2} - \ln \frac{1}{\sqrt{2}} + \ln \frac{3}{2} = \ln \left(\frac{3}{2} \cdot \frac{3}{2} \cdot 16\right) = \ln \frac{3}{2}
\]

15. \( \int \frac{x^3 + 3}{x^2 - 8x} \, dx = \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{x^2} \Rightarrow I = A(t + 2)(t - 1) + B(t - 1) + Ct(t + 2); t = 0 \Rightarrow A = -\frac{1}{6}; t = -2 \Rightarrow B = \frac{1}{6}; C = \frac{5}{16}; \int \frac{dt}{t^2 + 2} = -\frac{1}{2} \int \frac{dt}{t} + \frac{1}{6} \int \frac{dt}{t^2} + \frac{5}{16} \int \frac{dt}{t}

= -\frac{1}{2} \ln |t| + \frac{1}{6} \ln |t + 2| + \frac{5}{16} \ln |t - 1| + C
\]

16. \( \int 2x + 2 \, dx \Rightarrow J = A(x + 1)(x - 1) \Rightarrow A = 3, A + B = 2 \Rightarrow A = 3, B = -1; \int_{0}^{1} \frac{dx}{x^2 + 2x + 1}

= \int_{0}^{1} (x - 2) \, dx + 3 \int_{0}^{1} \frac{dx}{x + 1} - \int_{0}^{1} \frac{dx}{x + 1} = \left[ \frac{x^2}{2} - 2x + 3 \ln |x + 1| + \frac{27}{4} \right]_{0}^{1}

= \left( \frac{1}{2} - 2 + 3 \ln 2 - \frac{1}{4} \right) = 3 \ln 2 - 2
\]

17. \( \int \frac{3x + 2}{x^2 + 1} \, dx \Rightarrow \frac{3x - 2}{(x^2 + 1)^2} \Rightarrow 3x - 2 = A(x + 1) + B

= Ax + (A + B) \Rightarrow A = 3, A + B = 2 \Rightarrow A = 3, B = -1; \int_{0}^{1} \frac{dx}{x^2 + 2x + 1}

= \int_{0}^{1} (x + 2) \, dx + 3 \int_{0}^{1} \frac{dx}{x + 1} - \int_{0}^{1} \frac{dx}{x + 1} = \left[ \frac{x^2}{2} + 2x + 3 \ln |x - 1| - \frac{1}{1 - x} \right]_{1}^{0}

= (0 + 3 \ln 1 - \frac{1}{1}) - \left( \frac{1}{2} - 3 \ln 2 - \frac{1}{1 - 2} \right) = 2 - 3 \ln 2
\]

19. \( \int \frac{dx}{x^2 - 1} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{x + 1} + \frac{D}{x - 1} \Rightarrow 1 = A(x + 1)(x - 1)^2 + B(x - 1)(x + 1)^2 + C(x - 1)^2 + D(x + 1)^2;

x = -1 \Rightarrow C = \frac{1}{4}; x = 1 \Rightarrow D = \frac{1}{4}; \text{ coefficient of } x^3 = A + B \Rightarrow A + B = 0; \text{ constant } = A - B + C + D

\Rightarrow A - B + C + D = 1 \Rightarrow A = -\frac{1}{2}; \text{ thus, } A = \frac{1}{2} \Rightarrow B = \frac{1}{2}; \int_{0}^{1} \frac{dx}{x^2 - 1}

= \frac{1}{2} \int \frac{dx}{x + 1} - \frac{1}{2} \int \frac{dx}{x - 1} = \frac{1}{2} \ln |x + 1| + \frac{1}{2} \ln |x - 1| + \frac{1}{2} \ln |x + 1| + \frac{1}{2} \ln |x - 1| + \frac{1}{2} \ln |x + 1| + C

20. \( \int \frac{x^2}{(x - 1)(x^2 + 2x + 1)} \, dx = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x^2 + 1} \Rightarrow x^2 = A(x - 1)^2 + B(x - 1)(x + 1) + C(x - 1); x = -1

\Rightarrow C = -\frac{1}{2}; x = 1 \Rightarrow A = \frac{1}{2}; \text{ coefficient of } x^2 = A + B \Rightarrow A + B = 1 \Rightarrow B = \frac{3}{4}; \int_{0}^{1} \frac{x^2 \, dx}{(x - 1)(x^2 + 2x + 1)}

= \frac{1}{2} \int \frac{dx}{x - 1} + \frac{3}{2} \int \frac{dx}{x + 1} = \frac{1}{2} \ln |x - 1| + \frac{3}{2} \ln |x + 1| + \frac{1}{2} \ln |x + 1| + C

21. \( \int \frac{1}{x^2 + 1} \, dx = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \Rightarrow 1 = A(x^2 + 1) + (Bx + C)(x + 1); x = -1 \Rightarrow A = \frac{1}{2}; \text{ coefficient of } x^2

= A + B \Rightarrow A + B = 0 \Rightarrow B = -\frac{1}{2}; \text{ constant } = A + C \Rightarrow A + C = 1 \Rightarrow C = \frac{1}{2}; \int_{0}^{1} \frac{dx}{(x + 1)(x^2 + 1)}

= \frac{1}{2} \int_{0}^{1} \frac{dx}{x^2 + 1} + \frac{1}{2} \int_{0}^{1} \frac{dx}{x + 1} = \left[ \frac{1}{2} \ln |x + 1| - \ln \left(\frac{x + 1}{x - 1}\right) \right]_{0}^{1}

= \left( \frac{1}{2} \ln 2 - \frac{1}{2} \ln 2 + 2 \tan^{-1} 1 \right) - \left( \frac{1}{2} \ln 1 - \frac{1}{2} \ln 1 + \frac{1}{2} \tan^{-1} 0 \right) = \frac{1}{2} \ln 2 + \frac{1}{2} \left(\frac{\pi}{2}\right) = \frac{\pi + 2 \ln 2}{8}

22. \( \int \frac{x^3 + 1}{x^2 + 1} \, dx = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \Rightarrow 3x^2 + t + 4 = A(t^2 + 1) + (Bt + C)t; t = 0 \Rightarrow A = 4; \text{ coefficient of } t^2\)

= A + B \Rightarrow A + B = 3 \Rightarrow B = -1; \text{ coefficient of } t = C \Rightarrow C = 1; \int_{0}^{1} \frac{3x^2 + 1}{x^2 + 1} \, dt
\[ \frac{y^2 + 2y + 1}{(y^2 + 1)^2} = \frac{Ay + B}{y^2 + 1} + \frac{Cy + D}{(y^2 + 1)} \Rightarrow y^2 + 2y + 1 = (A + B)y^2 + Cy + D \]

\[ \int \frac{y^2 + 2y + 1}{(y^2 + 1)^2} \, dy = \int \frac{2}{y^2 + 1} \, dy + 2 \int \frac{1}{(y^2 + 1)} \, dy = \tan^{-1} y - \frac{1}{y^2 + 1} + C \]

\[ \frac{8x^2 + 8x + 2}{4(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)} \Rightarrow 8x^2 + 8x + 2 = (Ax + B)(4x^2 + 1) + Cx + D \]

\[ \int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2} \, dx = 2 \int \frac{dx}{x^2 + 1} + 8 \int \frac{dx}{(4x^2 + 1)^2} = \tan^{-1} 2x - \frac{1}{4x^2 + 1} + C \]

\[ \frac{s^4 + 81}{s^4 + 9} = \frac{A}{s + 3} + \frac{B + C}{s^2 + 9} + \frac{D + E}{s^3 + 9} \Rightarrow s^4 + 81 = A(s^2 + 9)^2 + (Bs + C)s(s^2 + 9) + (Ds + E)s \]

\[ \int \frac{s^4 + 81}{s^4 + 9} \, ds = \int \frac{ds}{s^2 + 1} - \int \frac{ds}{(s + 3)} + 2 \int \frac{ds}{(s^2 + 9)} = (-s - 1)^{-2} + (s - 1)^{-1} + \tan^{-1} s + C \]

\[ \int \frac{x^2 + 2}{x + 1} \, dx = \left[ u = x + 1 \Rightarrow u - 1 = x \Rightarrow du = dx \right] \]

\[ \int \frac{x^2 + 2}{x + 1} \, dx = \int \frac{u - 1}{u + 2} \, du = \frac{1}{2} \int \frac{u - 1}{u + 2} \, du \]

\[ = \frac{3}{2} \int \frac{u - 1}{u + 2} \, du = \frac{3}{2} \int \frac{u - 1}{u + 2} \, du = \frac{3}{2} \int \frac{1}{u + 4} \, du = \frac{3}{2} \int \frac{1}{u + 4} \, du \]

\[ = \frac{3}{2} \ln |x - 1| + \frac{6}{2} \ln |x + 1| + \frac{3}{4} - \frac{3}{\sqrt{3}} \tan^{-1} \left( \frac{x + 1}{\sqrt{3}/2} \right) + C = \frac{3}{2} \ln |x - 1| + \frac{3}{2} \ln |x + 1| - \sqrt{3} \tan^{-1} \left( \frac{2x + 1}{\sqrt{3}} \right) + C \]
28. \[
\frac{1}{x^2 + x + 1} = \frac{A}{x + 1} + \frac{B}{x + 2}\]

\[A = 2, B = 1\]

\[
\int \frac{1}{x^2 + x + 1} \, dx = \int \left( \frac{1}{x + 1} - \frac{1}{x + 2} \right) \, dx
\]

\[= \ln |x| - \frac{1}{2} \ln |x + 1| - \frac{1}{2} \ln |x^2 + x + 1| + C\]

29. \[
\frac{x^2}{x^2 + 3x - 4} = \frac{A}{x + 2} + \frac{B}{x - 2}
\]

\[\int \frac{x^2}{x^2 + 3x - 4} \, dx = \int \left( \frac{3}{x - 2} - \frac{1}{x + 2} \right) \, dx
\]

\[= 3 \ln |x - 2| - \ln |x + 2| + \frac{1}{2} \ln |x^2 + 1| + \frac{1}{2} \tan^{-1} x + C\]

30. \[
\frac{x^2 + 5x + 4}{x^2 + 2x + 2} = \frac{A}{x + 1} + \frac{B}{x + 2} + \frac{C}{x + 3}
\]

\[\int \frac{x^2 + 5x + 4}{x^2 + 2x + 2} \, dx = \int \left( \frac{x + 1}{x + 1} + \frac{2}{x + 2} + \frac{1}{x + 3} \right) \, dx
\]

\[= x + \ln |x + 2| + \frac{1}{2} \ln |x + 3| + C\]

31. \[
\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta + 1)^2} = \frac{A}{\theta + 1} + \frac{B}{(\theta + 1)^2} + \frac{C}{\theta + 2} + \frac{D}{\theta + 3}
\]

\[\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta + 1)^2} \, d\theta
\]

\[= \int \frac{\theta^4 + \theta^3 + \theta^2 + \theta + 1}{(\theta + 1)^2} \, d\theta
\]

\[= \frac{1}{3} \theta^3 + \frac{1}{2} \theta^2 + \frac{1}{3} \theta + \frac{1}{6} + C\]

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35. \( \frac{9x^2 - 3x + 1}{x^2 - x} = 9 + \frac{9x^2 - 3x + 1}{x(x - 1)} \) (after long division); \( \frac{9x^2 - 3x + 1}{x^2 - x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-1} \)

\[ \Rightarrow 9x^2 - 3x + 1 = Ax(x - 1) + B(x - 1) + Cx^2; \]

\[ x = 1 \Rightarrow C = 7; x = 0 \Rightarrow B = -1; A + C = 9 \Rightarrow A = 2; \]

\[ \int \frac{9x^2 - 3x + 1}{x^2 - x} \, dx = \int 9 \, dx + 2 \int \frac{dx}{x - 1} + 7 \int \frac{dx}{x + 1} = 9x + 2 \ln |x| + \frac{1}{x} + 7 \ln |x - 1| + C \]

36. \( \frac{16x^4}{4x^2 - 4x + 1} = (4x + 4)^2 \frac{12x - 4}{4x^2 - 4x + 1}; \frac{12x - 4}{2(2x - 1)^2} = \frac{A}{2x - 1} + \frac{B}{(2x - 1)^2} \Rightarrow 12x - 4 = A(2x - 1) + B \)

\[ \Rightarrow A = 6; -A + B = -4 \Rightarrow B = 2; \int \frac{16x^4}{4x^2 - 4x + 1} \, dx = 4 \int (x + 1) \, dx + 6 \int \frac{dx}{2x - 1} + 2 \int \frac{dx}{(2x - 1)^2} \]

\[ = 2(x + 1)^2 + 3 \ln |2x - 1| - \frac{1}{2x - 1} + C_1 = 2x^2 + 4x + 3 \ln |2x - 1| - (2x - 1)^{-1} + C, \text{ where } C = 2 + C_1 \]

37. \( \frac{y^2 + y - 1}{y^2 + y + 1}; \frac{1}{y^2 + y + 1} = \frac{A}{y} + \frac{By + C}{y^2 + y + 1} \Rightarrow 1 = A(y^2 + 1) + (By + C)y = (A + B)y^2 + Cy + A \)

\[ 7 \Rightarrow A = 1; A + B = 0 \Rightarrow B = -1; C = 0; \int \frac{y^2 + y - 1}{y^2 + y + 1} \, dy = \int y \, dy - \int \frac{dy}{y + 1} + \int \frac{y \, dy}{y^2 + 1} = \frac{y^2}{2} - \ln |y| + \frac{1}{2} \ln (1 + y^2) + C \]

38. \( \frac{2y^2}{y^2 + y - 1} = 2y + 2 \frac{1}{y^2 + y - 1}; \frac{1}{y^2 + y - 1} = \frac{2}{(y^2 + 1)(y + 1)} = \frac{A}{y + 1} + \frac{By + C}{y^2 + 1} \)

\[ \Rightarrow 2 = A(y^2 + 1) + (By + C)(y - 1) = (Ay^2 + A) + (By^2 + Cy - By - C) = (A + B)y^2 + (-B + C)y + (A - C) \]

\[ \Rightarrow A + B = 0, -B + C = 0 \text{ or } B = C, A - C = A - B = 2 \Rightarrow A = 1, B = -1, C = -1; \]

\[ \int \frac{2y^2}{y^2 + y - 1} \, dy = 2 \int (y + 1) \, dy + \int \frac{dy}{y + 1} - \int \frac{dy}{y^2 + 1} = (y + 1)^2 + \ln |y - 1| - \frac{1}{2} \ln (y^2 + 1) - \tan^{-1} y + C, \]

\[ \text{where } C = C_1 + 1 \]

39. \( \int \frac{e^t \, dt}{e^t + 3e^t + 2} = [e^t = y] \int \frac{dy}{y^2 + 3y + 2} = \int \frac{dy}{y + 1} - \int \frac{dy}{y + 2} = \ln \left| \frac{y + 1}{y + 2} \right| + C = \ln \left( \frac{e^{t+1}}{e^{t+2}} \right) + C \]

40. \( \int \frac{e^t + 2e^{3t} - e^t \, dt}{e^t + e^{3t} + 2} = \int \frac{e^t + 2e^{3t} - e^t}{e^t + e^{3t} + 2} \, e^t \, dt; \quad \left[ \begin{array}{c} \frac{y}{e^t} = e^t \\ \frac{dy}{e^t} = e^t \, dt \end{array} \right] \rightarrow \int \int \frac{y + 1}{y^2 + 1} \, dy = \frac{y^2}{2} + \int \frac{y \, dy}{y^2 + 1} \, dy - \int \frac{dy}{y^2 + 1} \]

\[ = \frac{x^2}{2} + \frac{1}{2} \ln (y^2 + 1) - \tan^{-1} y + C = \frac{1}{2} x^2 + \frac{1}{2} \ln (e^{2x} + 1) - \tan^{-1} (e^x) + C \]

41. \( \int \frac{\cos y \, dy}{\sin y + \sin y \cdot e^{-y}} \); \[ \sin y = t, \cos y \, dy = dt \rightarrow \int \frac{dy}{e^t + e^{-t} - 1} = \frac{1}{2} \int \frac{1}{1 - e^{-2t}} \, dt = \frac{1}{4} \int \frac{1}{1 - e^{-2t}} \, dt \]

\[ = \frac{1}{2} \ln \left| \frac{e^{-t}}{1 + e^{-2t}} \right| + C \]

42. \( \int \frac{\sin \theta \, d\theta}{\cos \theta + \cos \theta - 1} \); \[ \cos \theta = y \rightarrow - \int \frac{dy}{y + y - 1} = \frac{1}{3} \int dy + \frac{1}{3} \int dy = \frac{1}{3} \ln \left| \frac{y + 2}{y - 1} \right| + C = \frac{1}{3} \ln \left( \frac{\cos \theta + 2}{\cos \theta - 1} \right) + C \]

\[ = \frac{1}{3} \ln \left( \frac{\cos \theta + 2}{1 - \cos \theta} \right) + C \]

43. \( \int \frac{(x - 3)^2 \tan^{-1} (2x) - 12x^2 - 3x}{(4x^2 + 1)(x - 2)^2} \, dx = \int 	an^{-1} (2x) \, dx - 3 \int \frac{x}{(4x^2 + 1)} \, dx \]

\[ = \frac{1}{2} \int \tan^{-1} (2x) \, d(\tan^{-1} (2x)) - 3 \int \frac{dx}{x - 2} - 6 \int \frac{dx}{(x - 2)^4} = \frac{(\tan^{-1} 2x)^2}{4} - 3 \ln |x - 2| + \frac{6}{x - 2} + C \]

44. \( \int \frac{(x + 1)^2 \tan^{-1} (x) + 9x^3 + 3x}{(9x + 1)(x + 1)^3} \, dx = \int \frac{(x + 1)^2 \tan^{-1} (x) + 3x}{(9x + 1)(x + 1)^2} \, dx \]

\[ = \frac{1}{2} \int \tan^{-1} (x) \, d(\tan^{-1} (x)) + \int \frac{dx}{x + 1} \]

\[ = \frac{1}{2} \int \tan^{-1} (x) \, d(\tan^{-1} (x)) + \int \frac{dx}{x + 1} \rightarrow \int \frac{2}{x^2 + 1} \, du; \]

\[ \frac{2}{u^2 + 1} = \frac{A}{u + 1} + \frac{B}{u - 1} \Rightarrow 2 = A(u - 1) + B(u + 1) = (A + B)u - A + B \Rightarrow A + B = 0, -A + B = 2 \]

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\( \Rightarrow B = 1 \Rightarrow A = -1; \int \frac{x^2}{e^2 - 1} \, dx = \int \left( \frac{1}{u - 1} - \frac{1}{u + 1} \right) \, du = -\int \frac{1}{u + 1} \, du + \int \frac{1}{u - 1} \, du = -\ln|u + 1| + \ln|u - 1| + C \)

\( \Rightarrow \ln \left| \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right| + C \)

46. \( \int \frac{1}{x^3 + 1} \, dx \) [Let \( x = 6 \Rightarrow dx = 6u \, du \)]

\( \int \frac{1}{x^3 + 1} \, dx = \int \frac{6u^3}{u^3 + 1} \, du = \int \frac{6u^3 - 6u^3}{u^3 - 1} \, du = 6u - \int \frac{6u}{u^3 - 1} \, du = 6u - 3 \ln|u + 1| + 3 \ln|u - 1| + C = 6x^{1/6} + 3 \ln \left| \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right| + C \)

47. \( \int \frac{\sqrt{x + 4}}{x + 9} \, dx \) [Let \( x + 9 = u^2 \Rightarrow dx = 2u \, du \)]

\( \int \frac{1}{x^3 + 1} \, dx = \int \frac{2u^2}{u^2 + 1} \, du = \int \frac{2u^2 - 2u^2}{u^2 - 1} \, du = 2 - \int \frac{2u}{u^2 - 1} \, du = 2 - \int \frac{2u}{u^2 - 1} \, du = 2 - \int \frac{2}{u + 1} \, du + \int \frac{2}{u - 1} \, du = 2 - \frac{1}{u + 1} + \frac{1}{u - 1} + C = 2 + \frac{1}{\sqrt{x + 4} + 1} + C \)

48. \( \int \frac{1}{x(x^2 + 1)} \, dx \) [Let \( u = x^2 \Rightarrow du = 2x \, dx \)]

\( \int \frac{1}{x(x^2 + 1)} \, dx = \int \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int \frac{1}{u + 1} \, du = \frac{1}{2} \ln|u + 1| + \frac{1}{2} \ln|u - 1| + C = \frac{1}{2} \ln \left| \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right| + C \)

49. \( \int \frac{1}{x(x^4 + 1)} \, dx \) [Let \( u = x^4 \Rightarrow du = 4x^3 \, dx \)]

\( \int \frac{1}{x(x^4 + 1)} \, dx = \frac{1}{4} \int \frac{1}{u + 1} \, du = \frac{1}{4} \int \frac{1}{u + 1} \, du = \frac{1}{4} \int \frac{1}{u + 1} \, du = \frac{1}{4} \ln|u + 1| + C = \frac{1}{4} \ln \left| \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right| + C \)

50. \( \int \frac{1}{x(x^4 + 1)} \, dx \) [Let \( u = x^4 \Rightarrow du = 4x^3 \, dx \)]

\( \int \frac{1}{x(x^4 + 1)} \, dx = \frac{1}{4} \int \frac{1}{u + 1} \, du = \frac{1}{4} \int \frac{1}{u + 1} \, du = \frac{1}{4} \int \frac{1}{u + 1} \, du = \frac{1}{4} \ln|u + 1| + C = \frac{1}{4} \ln \left| \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right| + C \)

51. \( (t^2 - 3t + 2) \, dt = 1; x = \int \frac{dt}{t - 2} - \int \frac{dt}{t - 1} = \ln \left| \frac{t - 2}{t - 1} \right| + C; \frac{t - 2}{t - 1} = Ce^t; t = 3 \) and \( x = 0 \)

\( \Rightarrow \frac{1}{2} = C \Rightarrow \frac{t - 2}{t - 1} = \frac{1}{2} e^t \Rightarrow x = \ln \left| \frac{t - 2}{t - 1} \right| = \ln |t - 2| - \ln |t - 1| + \ln 2 \)

52. \( (3t^4 + 4t^2 + 1) \, dt = 2\sqrt{3}; x = 2\sqrt{3} \int \frac{dt}{t + \sqrt{3} + 1} = 2\sqrt{3} \int \frac{dt}{t + \sqrt{3} + 1} = \sqrt{3} \int \frac{dt}{t + \sqrt{3} + 1} - \sqrt{3} \int \frac{dt}{t + \sqrt{3} + 1} = 3 \tan^{-1} \left( \sqrt{3}t \right) - \sqrt{3} \tan^{-1} t + C; t = 1 \) and \( x = \frac{-3\sqrt{3}}{4} \Rightarrow x = -\frac{3\sqrt{3}}{4} \cdot \pi = -\frac{3\sqrt{3}}{4} \cdot \pi + C \Rightarrow C = -\pi \)

\( \Rightarrow x = 3 \tan^{-1} \left( \sqrt{3}t \right) - \sqrt{3} \tan^{-1} t - \pi \)
53. \( t^2 + 2t \frac{dx}{dt} = 2x + 2; \int_{t=1}^{x=1} \frac{dx}{t+1} = \int_{t=1}^{x=1} \frac{dt}{t+1} \Rightarrow 2 \ln |x + 1| = \ln |t + 1| + C; \) \\
\( t = 1 \) and \( x = 1 \) \( \Rightarrow \ln 2 = \ln \frac{1}{2} + C \Rightarrow C = \ln 2 \) \( + \ln 3 = 6 \Rightarrow \ln |x + 1| = \ln \frac{25}{12} \Rightarrow x + 1 = \frac{6}{12} \) \( t = 1, t > 0 \)

54. \( (t + 1) \frac{dx}{dt} = x^2 + 1 \Rightarrow \int_{t=1}^{x=1} \frac{dx}{t+1} \Rightarrow \tan^{-1} x = \ln |t + 1| + C; t = 0 \) and \( x = 0 \) \( \Rightarrow \tan^{-1} 0 = \ln |1| + C \) \( \Rightarrow C = \tan^{-1} 0 = 0 \Rightarrow \tan^{-1} x = \ln |t + 1| \Rightarrow x = \tan(\ln (t + 1)), t > -1 \)

55. \( V = \int_{x=1}^{y=1} \frac{dx}{x^3 + 2x^2} = \int_{x=1}^{y=1} \frac{dx}{x^3 + 2x^2} = 3 \pi \left( \int_{x=1}^{y=1} \frac{dx}{x^3} \right) = 3 \pi \ln |x + 1| \approx 25 \pi \ln 25 \)

56. \( V = 2\pi \int_{x=1}^{y=1} xy dx = 2\pi \int_{x=1}^{y=1} \frac{dx}{x + 1} = \left( \frac{\pi}{2} \right) \left( \frac{x}{x + 1} + 2 \ln |2 - x| \right) \approx \frac{4\pi}{3} \ln (2) \)

57. \( A = \int_{x=1}^{y=1} \tan^{-1} x dx = [x \tan^{-1} x]_{0}^{\pi} - \int_{0}^{\pi} \frac{dx}{x + \pi} \) \( = \frac{\pi}{2} \left[ \ln (x + 1) \right]_{0}^{\pi} = \frac{\pi}{2} \ln 3; \)
\( \bar{x} = \frac{1}{A} \int_{x=1}^{y=1} x \tan^{-1} x dx \)
\( = \frac{1}{A} \left( [\frac{1}{2} x^2 \tan^{-1} x]_{0}^{\pi} - \frac{1}{2} \int_{0}^{\pi} \frac{dx}{x + \pi} \right) \)
\( = \frac{1}{A} \left( \frac{\pi}{2} - \frac{1}{2} \left( \frac{\pi}{2} - \frac{\pi}{2} \right) \right) \approx 1.10 \)

58. \( A = \int_{x=1}^{y=1} x^2 + 13x - 9 dx = 3 \int_{x=1}^{y=1} \frac{dx}{x} - \int_{x=1}^{y=1} 2 \int_{x=1}^{y=1} \frac{dx}{x^2} + 2 \int_{x=1}^{y=1} \frac{dx}{x^3} = 3 \ln |x| - \ln |x + 3| + 2 \ln |x - 1| \approx \ln \frac{125}{9}; \)
\( \bar{x} = \frac{1}{A} \int_{x=1}^{y=1} x (x^2 + 13x - 9) dx = \frac{1}{A} \left( [4x^3]_{0}^{\pi} + 3 \int_{0}^{\pi} \frac{dx}{x} + 2 \int_{0}^{\pi} \frac{dx}{x^2} \right) = \frac{1}{A} (8 + 11 \ln 2 - 3 \ln 6) \approx 3.90 \)

59. (a) \( \frac{dx}{dt} = kx(N - x) \Rightarrow \int_{x=1}^{y=1} \frac{dx}{x(N - x)} = \int_{k=1}^{y=1} \frac{dt}{t} \Rightarrow \frac{1}{N} \int_{x=1}^{y=1} \frac{dx}{x} + \frac{1}{N} \int_{x=1}^{y=1} \frac{dx}{N-x} = \int_{k=1}^{y=1} \frac{dt}{t} \Rightarrow \frac{1}{N} \ln \frac{N-x}{x} = kt + C; \)
\( k = \frac{1}{N}, N = 1000, t = 0 \) and \( x = 2 \Rightarrow \frac{1}{1000} \ln \frac{1000-2}{2} = C \Rightarrow \frac{1}{1000} \ln \frac{1000-2}{2} = \frac{1}{1000} + \frac{1}{1000} \ln \left( \frac{1000}{2} \right) \)
\( \Rightarrow \ln \left( \frac{499x}{1000-x} \right) = 4t \Rightarrow \frac{499x}{1000-x} = e^{4t} \Rightarrow x = e^{4t} - 1000e^{4t} \Rightarrow x = 1000e^{4t} \Rightarrow t = \frac{1}{4} \ln 499 \approx 1.55 \text{ days} \)

60. \( \frac{dx}{dt} = k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)} = k dt \)
\( (a) a = b: \int_{x=1}^{y=1} \frac{dx}{(a-x)^2} = \int_{k=1}^{y=1} k dt \Rightarrow \frac{1}{a-x} = kt + C; t = 0 \) and \( x = 0 \Rightarrow \frac{1}{a} = C \Rightarrow \frac{1}{a-x} = kt + \frac{1}{a} \)
\( \Rightarrow \frac{1}{a-x} = \frac{ekt + 1}{a} \Rightarrow a-x = \frac{ekt + 1}{a} \Rightarrow x = a - \frac{ekt + 1}{a} \)
\( (b) a \neq b: \int_{x=1}^{y=1} \frac{dx}{(a-x)(b-x)} = \int_{k=1}^{y=1} k dt \Rightarrow \frac{1}{b-a} \int_{x=1}^{y=1} \frac{dx}{a-x} - \frac{1}{b-a} \int_{x=1}^{y=1} \frac{dx}{b-x} = \int_{k=1}^{y=1} \frac{dt}{b-a} \Rightarrow \frac{1}{b-a} \ln \left| \frac{b-x}{a-x} \right| = kt + C; \)
\( t = 0 \) and \( x = 0 \Rightarrow \frac{1}{b-a} \ln 1 = C \Rightarrow \ln \left| \frac{b-x}{a-x} \right| = (b-a)kt + \ln \left( \frac{b}{a} \right) \Rightarrow \frac{b-x}{a-x} = e^{(b-a)kt} \)
\( \Rightarrow x = \frac{ab - be^{(b-a)kt}}{a - be^{(b-a)kt}} \)

8.5 INTEGRAL TABLES AND COMPUTER ALGEBRA SYSTEMS

1. \( \int \frac{dx}{x\sqrt{x-3}} = \frac{2}{\sqrt{3}} \tan^{-1} \sqrt{\frac{x-3}{3}} + C \)
(We used FORMULA 13(a) with \( a = 1, b = 3 \))
2. \[
\int \frac{dx}{x\sqrt{x^4+4}} = \frac{1}{3} \ln \left| \frac{x + \sqrt{x^4 + 4}}{\sqrt{x^4 + 4} - \sqrt{x^4 + 2}} \right| + C
\]
(We used FORMULA 13(b) with \(a = 1, b = 4\))

3. \[
\int \frac{x^2}{\sqrt{x^2-2}} \, dx = \int \frac{(x^2-2) \, dx}{\sqrt{x^2-2}} + 2 \int \frac{dx}{\sqrt{x^2-2}} = \int \frac{(\sqrt{x^2-2})'}{\sqrt{x^2-2}} \, dx + 2 \int \frac{dx}{\sqrt{x^2-2}} = (\frac{1}{2}) \frac{(\sqrt{x^2-2})^3}{3} + 2 \ln \left| \sqrt{x^2-2} + \frac{1}{2} \right| + C
\]
(We used FORMULA 11 with \(a = 1, b = -2, n = 1\) and \(a = 1, b = 2, n = -1\))

4. \[
\int \frac{dx}{(2x+3)^3} = \frac{1}{2} \int \frac{(2x+3) \, dx}{(2x+3)^3} - \frac{3}{2} \int \frac{dx}{(2x+3)^2} = \frac{1}{2} \int \frac{dx}{\sqrt{2x+3}} - \frac{3}{2} \int \frac{dx}{(\sqrt{2x+3})^3} = \frac{1}{\sqrt{2x+3}} + C = \frac{(2x+3)^{-3/2}}{\sqrt{2x+3}} + C
\]
(We used FORMULA 11 with \(a = 2, b = 3, n = -1\) and \(a = 2, b = 3, n = -3\))

5. \[
\int x \, \sqrt{x^2 - 3} \, dx = \frac{1}{2} \int (2x^3 - 3 \, dx) + \frac{3}{2} \int \sqrt{x^2 - 3} \, dx = \frac{1}{2} \int \left( \frac{2x^3}{3} \right) \frac{(\sqrt{x^2-3})^3}{3} + C = \frac{(2x^3)^{1/2}}{2} \left[ \frac{x^3}{3} + 1 \right] + C = \frac{(2x^3)^{1/2}}{3} (x + 3) + C
\]
(We used FORMULA 11 with \(a = 2, b = -3, n = 3\) and \(a = 2, b = 3, n = -1\))

6. \[
\int x(7x + 5)^{3/2} \, dx = \frac{1}{7} \int (7x + 5)(7x + 5)^{3/2} \, dx - \frac{5}{7} \int (7x + 5)^{3/2} \, dx = \frac{1}{7} \int \sqrt{7x + 5} \frac{(7x + 5)^3}{3} + C = \left[ \left( \frac{7x + 5}{4} \right)^{3/2} \right] \left[ \frac{7x + 5}{3} - 2 \right] + C
\]
(We used FORMULA 11 with \(a = 7, b = 5, n = 5\) and \(a = 7, b = 5, n = 3\))

7. \[
\int \frac{dx}{(x^9 - 4x)^{3/2}} = -\frac{1}{(4x^{1/2})^3} + \frac{1}{2} \int \frac{dx}{x\sqrt{9x - 4x}} + C
\]
(We used FORMULA 14 with \(a = -4, b = 9\))

8. \[
\int \frac{dx}{x^2\sqrt{4x - 9}} = -\frac{1}{x^2} + 2 \int \frac{dx}{x\sqrt{4x - 9}} + C
\]
(We used FORMULA 15 with \(a = 4, b = -9\))

9. \[
\int x \sqrt{4x - x^2} \, dx = \int x \sqrt{2x - x^2} \, dx = \frac{1}{2} \int \sqrt{2x - x^2} \, dx = \frac{1}{3} \int \sqrt{2x - x^2} \, dx + \frac{1}{2} \int \sin^{-1} \left( \frac{x}{2} \right) + C
\]
(We used FORMULA 51 with \(a = 2\))
10. \[ \int \sqrt{\frac{2 + x - x^2}{x}} \, dx = \int \sqrt{2 \cdot \frac{1}{2} x - x^2 + \frac{1}{2} \sin^{-1} \left( \frac{x - \frac{1}{2}}{\frac{\sqrt{2}}{2}} \right)} + C = \sqrt{x - x^2} + \frac{1}{2} \sin^{-1} (2x - 1) + C \]
   (We used FORMULA 52 with \( a = \frac{1}{2} \))

11. \[ \int \frac{dx}{x \sqrt{7 + x^2}} = \int \frac{dx}{x \sqrt{\left(\sqrt{7}\right)^2 + x^2}} = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{\sqrt{7} + \sqrt{7 + x^2}}}{x} \right| + C = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{\sqrt{7} + \sqrt{7 + x^2}}}{x} \right| + C \]
   (We used FORMULA 26 with \( a = \sqrt{7} \))

12. \[ \int \frac{dx}{x \sqrt{7 - x^2}} = \int \frac{dx}{x \sqrt{\left(\sqrt{7}\right)^2 - x^2}} = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{\sqrt{7} - 
\sqrt{7 - x^2}}}{x} \right| + C = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{\sqrt{7} - 
\sqrt{7 - x^2}}}{x} \right| + C \]
   (We used FORMULA 34 with \( a = \sqrt{7} \))

13. \[ \int \frac{\sqrt{x^2 - 2x}}{x} \, dx = \int \sqrt{2^2 - x^2} - 2 \ln \left| \frac{\sqrt{2 + \sqrt{2^2 - x^2}}}{x} \right| + C = \sqrt{x^2 - 4} - 2 \ln \left| \frac{\sqrt{\sqrt{2} + \sqrt{4 - x^2}}}{x} \right| + C \]
   (We used FORMULA 31 with \( a = 2 \))

14. \[ \int \frac{\sqrt{x^2 - 4}}{x} \, dx = \int \sqrt{x^2 - 2^2} - 2 \sec^{-1} \left| \frac{\sqrt{\frac{x}{2}}}{x} \right| + C = \sqrt{x^2 - 4} - 2 \sec^{-1} \left| \frac{\sqrt{\frac{x}{2}}}{x} \right| + C \]
   (We used FORMULA 42 with \( a = 2 \))

15. \[ \int e^{2t} \cos 3t \, dt = \frac{e^{2t}}{2^2 + 3^2} (2 \cos 3t + 3 \sin 3t) + C = \frac{e^{2t}}{13} (2 \cos 3t + 3 \sin 3t) + C \]
   (We used FORMULA 108 with \( a = 2, b = 3 \))

16. \[ \int e^{-3t} \sin 4t \, dt = \frac{-e^{-3t}}{-3^2 + 4^2} (-3 \sin 4t - 4 \cos 4t) + C = \frac{e^{-3t}}{25} (-3 \sin 4t - 4 \cos 4t) + C \]
   (We used FORMULA 107 with \( a = -3, b = 4 \))

17. \[ \int x \cos^{-1} x \, dx = \int x^{\cos^{-1} x} \, dx = \frac{x^{\cos^{-1} x} \cos^{-1} x}{\cos^{-1} x} + \frac{1}{\cos^{-1} x} \int \frac{x^{\cos^{-1} x} \, dx}{\sqrt{1 - x^2}} = \frac{x^{\cos^{-1} x} \cos^{-1} x}{\cos^{-1} x} + \frac{1}{\cos^{-1} x} \int \frac{x^{\cos^{-1} x} \, dx}{\sqrt{1 - x^2}} \]
   (We used FORMULA 100 with \( a = 1, n = 1 \))
   \[ = \frac{x^{\cos^{-1} x}}{\cos^{-1} x} + \frac{1}{\sqrt{2}} \frac{1}{\sin^{-1} x} - \frac{1}{\sqrt{2}} \left( \frac{1}{\sin^{-1} x} \right) x \sqrt{1 - x^2} + C = \frac{x^{\cos^{-1} x}}{\cos^{-1} x} + \frac{1}{\sqrt{2}} \sin^{-1} x - \frac{1}{\sqrt{2}} x \sqrt{1 - x^2} + C \]
   (We used FORMULA 33 with \( a = 1 \))

18. \[ \int x \tan^{-1} x \, dx = \int x \tan^{-1} (1x) \, dx = \frac{x^{\tan^{-1} (1x)}}{1 + 1} \tan^{-1} (1x) - \frac{1}{1 + 1} \int \frac{x^{\tan^{-1} (1x)} \, dx}{1 + (1x)^2} = \frac{x^{\tan^{-1} x}}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^{\tan^{-1} x} \, dx}{1 + x^2} \]
   (We used FORMULA 101 with \( a = 1, n = 1 \))
   \[ = \frac{x^{\tan^{-1} x}}{2} \tan^{-1} x - \frac{1}{2} \int \left( \frac{1}{1 + x^2} \right) dx \text{ (after long division)} \]
   \[ = \frac{x^{\tan^{-1} x}}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{1 + x^2} dx = \frac{x^{\tan^{-1} x}}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C = \frac{1}{2} ((x^2 + 1)\tan^{-1} x - x) + C \]

19. \[ \int x^2 \tan^{-1} x \, dx = \frac{x^{2+1}}{2+1} \tan^{-1} x - \frac{1}{2+1} \int \frac{x^{2+1}}{1 + x^2} \, dx = \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3 \, dx}{1 + x^2} \]
   (We used FORMULA 101 with \( a = 1, n = 2 \);)
   \[ \int \frac{x^3 \, dx}{1 + x^2} = \int x \, dx - \int \frac{x \, dx}{1 + x^2} = \frac{x^2}{2} - \frac{1}{2} \ln (1 + x^2) + C \Rightarrow \int x^2 \tan^{-1} x \, dx \]
   \[ = \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln (1 + x^2) + C \]
20. \[ \int \tan^{-1} x \, dx = \int x^{-2} \tan^{-1} x \, dx = \frac{\tan^{-1} x}{-2 + x} \tan^{-1} x - \frac{1}{2} \int \frac{1}{1+x^2} \, dx = \frac{x^{-1}}{1} \tan^{-1} x + \int \frac{x^{-1}}{(1+x^2)} \, dx \]

(We used FORMULA 101 with \( a = 1, \, n = -2 \));

\[ \int \frac{x \, dx}{1+x^2} = \int \frac{dx}{x(1+x^2)} = \int \frac{dx}{x} - \int \frac{x \, dx}{1+x^2} = \ln |x| - \frac{1}{2} \ln (1 + x^2) + C \]

\[ \Rightarrow \int \frac{\tan^{-1} x \, dx}{x^2} = \frac{1}{x} \tan^{-1} x + \ln |x| - \frac{1}{2} \ln (1 + x^2) + C \]

21. \[ \int \sin 3x \cos 2x \, dx = \frac{-\cos 5x}{10} - \frac{\cos x}{2} + C \]

(We used FORMULA 62(a) with \( a = 3, \, b = 2 \))

22. \[ \int 2x \cos 3x \, dx = -\frac{\cos 5x}{10} + \frac{\cos x}{2} + C \]

(We used FORMULA 62(a) with \( a = 2, \, b = 3 \))

23. \[ \int 8 \sin 4t \sin \left( \frac{\pi}{4} t \right) \, dt = \frac{8}{\pi} \sin \left( \frac{\pi}{8} \right) + \frac{8}{\pi} \sin \left( \frac{7\pi}{8} \right) + C = 8 \left[ \frac{\sin \left( \frac{\pi}{4} \right)}{\pi} - \frac{\sin \left( \frac{7\pi}{8} \right)}{\pi} \right] + C \]

(We used FORMULA 62(b) with \( a = 4, \, b = 1/2 \))

24. \[ \int \sin \left( \frac{\pi}{6} \right) \sin \left( \frac{\pi}{5} \right) \, dt = 3 \sin \left( \frac{\pi}{5} \right) - \sin \left( \frac{\pi}{6} \right) + C \]

(We used FORMULA 62(b) with \( a = 1/6, \, b = 1/5 \))

25. \[ \int \cos \left( \frac{\pi}{6} \right) \cos \left( \frac{\pi}{7} \right) \, d\theta = 6 \sin \left( \frac{\pi}{12} \right) + \frac{6}{\pi} \sin \left( \frac{7\pi}{12} \right) + C \]

(We used FORMULA 62(c) with \( a = 1/6, \, b = 1/7 \))

26. \[ \int \cos \left( \frac{\pi}{6} \right) \cos \left( \frac{\pi}{7} \right) \, d\theta = 1/12 \sin \left( \frac{13\pi}{12} \right) + \frac{1}{15} \sin \left( \frac{15\pi}{12} \right) + C = \frac{\sin \left( \frac{13\pi}{13} \right)}{13} + \frac{\sin \left( \frac{15\pi}{15} \right)}{15} + C \]

(We used FORMULA 62(c) with \( a = 1/6, \, b = 7 \))

27. \[ \int \frac{x^2 + x + 1}{(x^2 + 1)^2} \, dx = \int \frac{x \, dx}{x^2 + 1} + \int \frac{dx}{(x^2 + 1)^2} = \frac{1}{2} \int \frac{d(x^2 + 1)}{x^2 + 1} + \int \frac{dx}{(x^2 + 1)^2} \]

\[ = \frac{1}{2} \ln(x^2 + 1) + \frac{x}{2(1 + x^2)} + \frac{1}{2} \tan^{-1} x + C \]

(For the second integral we used FORMULA 17 with \( a = 1 \))

28. \[ \int \frac{x^2 + 6x}{(x^2 + 3)^2} \, dx = \int \frac{dx}{x^2 + 3} + \int \frac{6x \, dx}{(x^2 + 3)^2} - \int \frac{3 \, dx}{(x^2 + 3)^2} = \int \frac{dx}{x^2 + (\sqrt{3})^2} + 3 \int \frac{d(x^2 + 3)}{(x^2 + 3)^2} - 3 \int \frac{dx}{x^2 + (\sqrt{3})^2} \]

\[ = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{x}{\sqrt{3}} \right) - \frac{3}{x^2 + 3} - 3 \left( \frac{x}{(x^2 + 3)^2} + \frac{1}{2\sqrt{3}} \tan^{-1} \left( \frac{x}{\sqrt{3}} \right) \right) + C \]

(For the first integral we used FORMULA 16 with \( a = \sqrt{3} \); for the third integral we used FORMULA 17 with \( a = \sqrt{3} \))

\[ = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{x}{\sqrt{3}} \right) - \frac{x}{x^2 + 3} - \frac{3x}{2(x^2 + 3)} + C \]

29. \[ \int \sin^{-1} \sqrt{x} \, dx; \quad \left[ u = \sqrt{x}, \quad x = u^2, \quad dx = 2u \, du \right] \Rightarrow 2 \int u \sin^{-1} u \, du = 2 \left( \frac{u^{n+1}}{n+1} \sin^{-1} u - \frac{1}{1+n} \int \frac{u^{n+1}}{\sqrt{1-u^2}} \, du \right) \]
\[ u^2 \sin^{-1} u - \int \frac{u^3 \, du}{\sqrt{1-u^2}} \]

(We used FORMULA 99 with a = 1, n = 1)

\[ u^2 \sin^{-1} u - \left( \frac{1}{2} \sin^{-1} u - \frac{1}{2} u \sqrt{1-u^2} \right) + C = (u^2 - \frac{1}{2}) \sin^{-1} u + \frac{1}{2} u \sqrt{1-u^2} + C \]

(We used FORMULA 33 with a = 1)

\[ (x - \frac{1}{2}) \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x-x^2} + C \]

30. \[ \int \frac{\cos^{-1} \sqrt{x}}{\sqrt{x}} \, dx; \quad \frac{u = \sqrt{x}}{dx = 2u \, du} \rightarrow \int \frac{\cos^{-1} u \cdot 2u \, du}{\sqrt{1-u^2}} = 2\int \cos^{-1} u \, du = 2 \left( u \cos^{-1} u - \frac{1}{2} \sqrt{1-u^2} \right) + C \]

(We used FORMULA 97 with a = 1)

\[ = 2 \left( \sqrt{x} \cos^{-1} \sqrt{x} - \sqrt{1-x} \right) + C \]

31. \[ \int \frac{\sqrt{x}}{\sqrt{1-x}} \, dx; \quad \frac{u = \sqrt{x}}{dx = 2u \, du} \rightarrow \int \frac{u^2 \, du}{\sqrt{1-u^2}} = 2\int \frac{u^2 \, du}{\sqrt{1-u^2}} = 2 \left( \frac{1}{2} \sin^{-1} u - \frac{1}{2} u \sqrt{1-u^2} \right) + C \]

(We used FORMULA 33 with a = 1)

\[ = \sin^{-1} u - u \sqrt{1-u^2} + C \]

32. \[ \int \frac{\sqrt{x^2-x}}{\sqrt{x}} \, dx; \quad \frac{u = \sqrt{x}}{dx = 2u \, du} \rightarrow \int \frac{\sqrt{u^2-u} \cdot 2u \, du}{\sqrt{1-u^2}} = 2\int \frac{\sqrt{u^2-u}}{\sqrt{1-u^2}} \, du = 2 \left( \frac{1}{2} \sin^{-1} u - \frac{1}{2} u \sqrt{1-u^2} \right) + C \]

(We used FORMULA 29 with a = 1)

\[ = \sqrt{2x-x^2} + 2 \sin^{-1} \sqrt{\frac{2}{2}} + C \]

33. \[ \int (\cot t) \sqrt{1 - \sin^2 t} \, dt = \int \frac{\sqrt{1 - \sin^2 (\cot t)} \, dt}{\sin t} \rightarrow \frac{u = \sin t \, du = \cos t \, dt}{\int \sqrt{1-u^2} \, du} \]

\[ = \sqrt{1-u^2} - \ln \left| \frac{1+\sqrt{1-u^2}}{u} \right| + C \]

(We used FORMULA 31 with a = 1)

\[ = \sqrt{1-\sin^2 t} - \ln \left| \frac{1+\sqrt{1-\sin^2 t}}{\sin t} \right| + C \]

34. \[ \int \frac{dt}{\left( \tan t \right) \sqrt{4 - \sin^2 t}} = \int \frac{\cos t \, dt}{\sin t \sqrt{4 - \sin^2 t}} \rightarrow \frac{u = \sin t \, du = \cos t \, dt}{\int \frac{du}{u \sqrt{4-u^2}}} = -\frac{1}{2} \ln \left| \frac{2+\sqrt{4-u^2}}{u} \right| + C \]

(We used FORMULA 34 with a = 2)

\[ = -\frac{1}{2} \ln \left| \frac{2+\sqrt{4-\sin^2 t}}{\sin t} \right| + C \]

35. \[ \int \frac{dy}{\sqrt{3 + (\ln y)^2}} \rightarrow \int \frac{e^t \, du}{\sqrt{3+u^2}} = \int \frac{du}{\sqrt{3+u^2}} = \ln \left| u + \sqrt{3+u^2} \right| + C \]

(We used FORMULA 20 with a = \sqrt{3})

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36. \(\int \tan^{-1} \sqrt{y} \, dy;\) \[ t = \sqrt{y}, \quad y = t^2 \] \[ \frac{1}{2} t \tan^{-1} t - \frac{1}{2} \int \frac{t}{1 + t^2} \, dt = t^2 \tan^{-1} t - \int \frac{t}{1 + t^2} \, dt \]

(We used FORMULA 101 with \(n = 1, a = 1\))

\[ t^2 \tan^{-1} t - \int \frac{t + 1}{1 + t^2} \, dt + \int \frac{dt}{1 + t^2} = t^2 \tan^{-1} t - t + \tan^{-1} t + C = y \tan^{-1} \sqrt{y} + \tan^{-1} \frac{\sqrt{y}}{y} + C \]

37. \(\int \frac{1}{\sqrt{x^2 + 2x + 5}} \, dx = \int \frac{1}{\sqrt{(x+1)^2 + 4}} \, dx;\) \[ t = x + 1, \quad dt = dx \] \[ \int \frac{1}{\sqrt{t^2 + 1}} \, dt = \int \frac{1}{\sqrt{t^2 + 1}} \, dt \]

(We used FORMULA 20 with \(a = 2\))

\[ \ln|t + \sqrt{t^2 + 4}| + C = \ln|x + 1 + \sqrt{(x+1)^2 + 4}| + C = \ln|x + 1 + \sqrt{x^2 + 2x + 5}| + C \]

38. \(\int \frac{x^2}{\sqrt{x^2 - 4x + 5}} \, dx = \int \frac{x^2}{\sqrt{(x-2)^2 + 1}} \, dx;\) \[ t = x - 2, \quad dt = dx \] \[ \int \frac{(t+2)^2}{\sqrt{t^2 + 1}} \, dt = \int \frac{t^2 + 4t + 2}{\sqrt{t^2 + 1}} \, dt = \int \frac{dt}{\sqrt{t^2 + 1}} + \int \frac{4t}{\sqrt{t^2 + 1}} \, dt \]

(We used FORMULA 25 with \(a = 1\))

\[ \frac{1}{2} \ln|t + \sqrt{t^2 + 1}| + C = \frac{1}{2} \ln|t + \sqrt{t^2 + 1}| + C \]

39. \(\int \sqrt{5 - 4x - x^2} \, dx = \int \sqrt{9 - (x + 2)^2} \, dx;\) \[ t = x + 2, \quad dt = dx \] \[ \int \sqrt{9 - t^2} \, dt \]

(We used FORMULA 29 with \(a = 3\))

\[ \frac{1}{2} \sqrt{9 - t^2} + \frac{3}{2} \sin^{-1} \left( \frac{t}{3} \right) + C = \frac{1}{2} \sqrt{9 - (x + 2)^2} + \frac{3}{2} \sin^{-1} \left( \frac{x + 2}{3} \right) + C = \frac{1}{2} \sqrt{5 - 4x - x^2} + \frac{3}{2} \sin^{-1} \left( \frac{x + 2}{3} \right) + C \]

40. \(\int x^2 \sqrt{2x - x^2} \, dx = \int x^2 \sqrt{1 - (x - 1)^2} \, dx;\) \[ t = x - 1, \quad dt = dx \] \[ \int (t + 1)^2 \sqrt{1 - t^2} \, dt = \int \left( \frac{t^2 + 2t + 1}{\sqrt{1 - t^2}} \right) \, dt \]

(We used FORMULA 30 with \(a = 1\))

\[ \frac{t}{4} \sin^{-1} \left( \frac{t}{2} \right) - \frac{1}{8} t \sqrt{1 - t^2} \left( 1 - 2t^2 \right) - \frac{1}{4} \left( 1 - t^2 \right)^{3/2} + \left[ \frac{3}{4} \right] \sqrt{1 - t^2} + \frac{3}{2} \sin^{-1} \left( \frac{t}{2} \right) + C \]

\[ \frac{1}{2} \sqrt{9 - t^2} + \frac{3}{8} \sin^{-1} \left( \frac{t}{3} \right) + \frac{1}{8} \sqrt{1 - t^2} \left( 2 - (x - 1)^2 \right) + \frac{3}{4} \left( 1 - (x - 1)^2 \right)^{3/2} + \frac{3}{2} \sqrt{2x - x^2} + \frac{3}{8} \sin^{-1} \left( \frac{x - 1}{2} \right) + C \]

41. \(\int \sin^3 2x \, dx = -\sin^3 2x \cos 2x + \frac{3}{5} \int \sin^3 2x \, dx = -\sin^3 2x \cos 2x + \frac{4}{5} \left[ -\sin^3 2x \cos 2x + \frac{3}{5} \int \sin 2x \, dx \right] \)

(We used FORMULA 60 with \(a = 2, n = 5\) and \(a = 2, n = 3\))

\[ -\sin^3 2x \cos 2x + \frac{4}{15} \sin^2 2x \cos 2x + \frac{8}{15} \left( -\frac{1}{2} \right) \cos 2x + C = -\sin^3 2x \cos 2x - \frac{2}{3} \sin^2 2x \cos 2x - \frac{4}{15} \cos 2x + C \]

42. \(\int 8 \cos^4 2\pi t \, dt = \left( \cos^2 2\pi t \sin 2\pi t + \frac{1}{4} \int \cos^2 2\pi t \, dt \right) \)

(We used FORMULA 61 with \(a = 2\pi, n = 4\))

\[ \frac{\cos^2 2\pi t \sin 2\pi t}{\pi} + 6 \left[ -\frac{1}{2} + \frac{\sin (2\pi t)}{4\pi} \right] + C \]

(We used FORMULA 59 with \(a = 2\pi\))

\[ \frac{\cos^2 2\pi t \sin 2\pi t}{\pi} + 3t + \frac{3}{4} \sin 2\pi t + C = \cos^2 2\pi t \sin 2\pi t + \frac{3}{2} \cos 2\pi t + 3t + C \]

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43. \[\int \sin^2 \theta \cos^2 \theta \, d\theta = \frac{\sin^2 \theta \cos^2 \theta}{2(2+3)} + \frac{3 - 1}{3} \int \sin^2 \theta \cos 2\theta \, d\theta\]

(We used FORMULA 69 with \(a = 2, m = 3, n = 2\))

\[= \frac{\sin^2 \theta \cos 2\theta}{10} + \frac{3}{5} \int \sin^2 \theta \cos 2\theta \, d\theta = \frac{\sin^2 \theta \cos^2 2\theta}{10} + \frac{3}{5} \left[ \frac{1}{2} \int \sin^2 \theta \, d(\sin 2\theta) \right] = \frac{\sin^2 \theta \cos^2 \theta}{10} + \frac{3\sin^2 \theta}{15} + C\]

44. \[\int 2 \sin^2 t \sec^4 t \, dt = \int 2 \sin^2 t \cos^{-4} t \, dt = 2 \left( -\frac{\sin t \cos^{-3} t}{2-4} + \frac{3 - 1}{2} \int \cos^{-4} t \, dt \right)\]

(We used FORMULA 68 with \(a = 1, n = 2, m = -4\))

\[= \sin t \cos^{-3} t - \int \cos^{-4} t \, dt = \sin t \cos^{-3} t - \int \sec^4 t \, dt = \sin t \cos^{-3} t - \left( \frac{\sec^2 t \tan t}{4-1} + \frac{3 - 2}{4-1} \int \sec^2 t \, dt \right)\]

(We used FORMULA 92 with \(a = 1, n = 4\))

\[= \sin t \cos^{-3} t - \left( \frac{\sec^2 t \tan t}{3} \right) - \frac{2}{3} \tan t + C = \frac{2}{3} \sec^2 t \tan t - \frac{2}{3} \tan t + C = \frac{2}{3} \tan (sec^2 t - 1) + C\]

\[= \frac{2}{3} \tan^3 t + C\]

An easy way to find the integral using substitution:

\[\int 2 \sin^2 t \cos^{-4} t \, dt = \int 2 \tan^2 t \sec^2 t \, dt = \int 2 \tan^2 t \, d(\tan t) = \frac{2}{3} \tan^3 t + C\]

45. \[\int 4 \tan^3 2x \, dx = 4 \left( \frac{\tan^2 2x}{2-2} - \int \tan 2x \, dx \right) = \tan 2x - 4 \int \tan 2x \, dx\]

(We used FORMULA 86 with \(n = 3, a = 2\))

\[= \tan 2x - \frac{\ln |\sec 2x| + C}{2} = \tan 2x - 2 \ln |\sec 2x| + C\]

46. \[\int 8 \cot^4 t \, dt = 8 \left( -\frac{\cot^3 t}{3} + \int \cot^2 t \, dt \right)\]

(We used FORMULA 87 with \(a = 1, n = 4\))

\[= 8 \left( -\frac{1}{3} \cot^3 t + \cot t + t \right) + C\]

(We used FORMULA 85 with \(a = 1\))

47. \[\int 2 \sec^3 \pi x \, dx = 2 \left[ \frac{\sec \pi x \tan \pi x}{\pi(3-1)} + \frac{3 - 2}{3} \int \sec \pi x \, dx \right]\]

(We used FORMULA 92 with \(n = 3, a = \pi\))

\[= \frac{2}{3} \sec \pi x \tan \pi x + \frac{2}{3} \ln |\sec \pi x + \tan \pi x| + C\]

(We used FORMULA 88 with \(a = \pi\))

48. \[\int 3 \sec^4 3x \, dx = 3 \left[ \frac{\sec^3 3x \tan 3x}{3(3-1)} + \frac{4 - 2}{4-1} \int \sec^2 3x \, dx \right]\]

(We used FORMULA 92 with \(n = 4, a = 3\))

\[= \frac{3}{3} \sec^3 3x \tan 3x + \frac{2}{3} \tan 3x + C\]

(We used FORMULA 90 with \(a = 3\))

49. \[\int \csc^3 x \, dx = -\frac{\csc^3 x \cot x}{5 - 3} + \frac{5 - 2}{5} \int \csc^3 x \, dx = -\frac{\csc^3 x \cot x}{3} + \frac{3}{4} \left( -\frac{\csc^3 x \cot x}{3} + \frac{3 - 2}{3} \int \csc x \, dx \right)\]

(We used FORMULA 93 with \(n = 5, a = 1\) and \(n = 3, a = 1\))

\[= -\frac{1}{3} \csc^3 x \cot x - \frac{2}{3} \csc x \cot x + \frac{3}{8} \ln |\csc x + \cot x| + C\]

(We used FORMULA 89 with \(a = 1\))

50. \[\int 16x^5(\ln x)^2 \, dx = 16 \left[ \frac{\ln x}{4} \right] - \frac{2}{3} \int x^3 \ln x \, dx \right] = 16 \left[ \frac{x^4(\ln x)^2}{4} - \frac{1}{2} \left( \frac{x^4(\ln x)}{4} - \frac{1}{4} \int x^3 \, dx \right) \right]\]

(We used FORMULA 110 with \(a = 1, n = 3, m = 2\) and \(a = 1, n = 3, m = 1\))

\[= 16 \left( \frac{x^4(\ln x)^2}{8} - \frac{x^4(\ln x)}{32} + \frac{x^4}{32} \right) + C = 4x^4(\ln x)^2 - 2x^4 \ln x + \frac{x^4}{2} + C\]
51. \[ \int e^{e^x} \sec^3 (e^x - 1) \, dx; \quad \text{where } x = e^{e^x - 1} \]

\[ \int \sec^3 x \, dx = \frac{\sec x \tan x}{3-1} + \frac{3-2}{3-1} \int \sec x \, dx \]

(We used FORMULA 92 with \( a = 1, n = 3 \))

\[ = \frac{\sec x \tan x}{3-1} + \frac{3-2}{3-1} \int \sec x \, dx \]

\[ = \frac{\sec x \tan x}{3-1} + \frac{3-2}{3-1} \] \( \ln |\sec x + \tan x| + C = \frac{1}{2} |\sec (e^x - 1) \tan (e^x - 1) + \ln |\sec (e^x - 1) + \tan (e^x - 1)|| + C \]

52. \[ \int \sec^3 \sqrt{\theta} \, d\theta; \quad \text{where } \theta = t^2 \]

\[ \int 2 \sec^3 t \, dt = 2 \left[ - \frac{\sec t \cot t}{3-2} + \frac{3-2}{3-1} \int \sec t \, dt \right] \]

(We used FORMULA 93 with \( a = 1, n = 3 \))

\[ = 2 \left[ - \frac{\sec t \cot t}{3-2} + \frac{3-2}{3-1} \right] + C \]

53. \[ \int_0^1 2 \sqrt{x^2 + 1} \, dx; \quad \text{where } x = \tan t \]

\[ = 2 \int_0^{\pi/4} \sec t \cdot \sec^2 t \, dt = 2 \int_0^{\pi/4} \sec^3 t \, dt = 2 \left[ \frac{\sec t \tan t}{3-1} \right]_0^{\pi/4} + \frac{3-2}{3-1} \int_0^{\pi/4} \sec t \, dt \]

(We used FORMULA 92 with \( n = 3, a = 1 \))

\[ = \frac{\sec t \tan t}{3-1} + \frac{3-2}{3-1} \] \( \ln \sqrt{2} + C \)

54. \[ \int_0^{\sqrt{2}/2} \frac{\cos x \, dx}{\sqrt{1 - x^2}} = \int_0^{\cos^2} \sec^2 x \, dx = \int_0^{\pi/3} \sec^4 x \, dx \]

(We used FORMULA 92 with \( a = 1, n = 4 \))

\[ = \frac{\sec^2 x \tan x}{3-3} + \frac{2}{3} \tan x \]

\[ = \left( \frac{1}{3} \right) \sqrt{3} + \left( \frac{2}{3} \right) \sqrt{3} = 2 \sqrt{3} \]

55. \[ \int_1^{\sqrt{2}} \frac{\tan \theta}{\sec \theta} \, d\theta; \quad \text{where } \theta = \tan \theta \]

\[ \int_1^{\pi/3} \tan^3 \theta \, d\theta = \int_1^{\pi/3} \sec^3 \theta \, d\theta = \int_1^{\tan \theta} \tan^3 \theta \, d\theta \]

\[ = \left[ \tan^3 \theta \right]_0^{\pi/3} - \int_0^{\tan \theta} \tan^2 \theta \, d\theta \]

\[ = \left( \frac{3\sqrt{3}}{3} \right) - \left( \frac{\pi}{3} \right) = \frac{3\sqrt{3}}{3} - \frac{\pi}{3} = \frac{\pi}{3} \]

(We used FORMULA 86 with \( a = 1, n = 4 \) and FORMULA 84 with \( a = 1 \))

56. \[ \int_0^{1/\sqrt{3}} \frac{dt}{(r^2 + 1)t^2}; \quad \text{where } r = \tan \theta \]

\[ = \int_0^{\tan^6 \theta} \sec^6 \theta \, d\theta \]

\[ = \int_0^{\cos^6 \theta} \cos^6 \theta \, d\theta \]

\[ = \frac{\cos^6 \theta \sin \theta}{5} + \frac{4}{3} \int_0^{\cos^6 \theta} \cos^6 \theta \, d\theta \]

\[ = \frac{\cos^6 \theta \sin \theta}{5} + \frac{4}{3} \int_0^{\cos^6 \theta} \cos^6 \theta \, d\theta \]

(We used FORMULA 61 with \( a = 1, n = 5 \) and \( a = 1, n = 3 \))

\[ = \left( \frac{9}{5} \right) \left( \frac{1}{2} \right) + \left( \frac{9}{10} \right) \]

\[ = \frac{9}{160} + \frac{4}{15} = \frac{3}{4} \frac{480}{324} + \frac{3}{4} \frac{480}{324} = \frac{203}{480} \]

57. \[ S = \int_0^{\sqrt{2}} 2\pi y \sqrt{1 + (y')^2} \, dx \]

\[ = 2\pi \int_0^{\sqrt{2}} \sqrt{\frac{y^2 + 2 + 1 + \frac{x^2}{y^2}}{2}} \, dx \]

\[ = 2\pi \int_0^{\sqrt{2}} \sqrt{x^2 + 1} \, dx \]

\[ = 2\pi \left[ \frac{x\sqrt{x^2 + 1}}{2} + \frac{1}{2} \ln \left| x + \sqrt{x^2 + 1} \right| \right]_0^{\sqrt{2}} \]

(We used FORMULA 21 with \( a = 1 \))

\[ = 2\pi \left[ \sqrt{6} + \ln (\sqrt{2} + \sqrt{3}) \right] = 2\pi \sqrt{3} + \pi \sqrt{2} \ln (\sqrt{2} + \sqrt{3}) \]

58. \[ L = \int_0^{\sqrt{2}/2} \sqrt{1 + (2x)^2} \, dx = \int_0^{\sqrt{2}/2} \sqrt{\frac{1 + x^2}{2}} \, dx = \int_0^{\sqrt{2}/2} \left[ \frac{1}{\sqrt{2}} \frac{x + \sqrt{x^2 + 1}}{2} \right] \ln (x + \sqrt{x^2 + 1})^2 \]

(We used FORMULA 2 with \( a = \frac{1}{2} \))
59. \( A = \int_{0}^{3} \frac{dx}{x+1} = \left[ 2\sqrt{x+1} \right]_{0}^{3} = 2; \quad x = \frac{1}{x} \int_{0}^{3} \frac{dx}{x+1} \)
\[
= \frac{1}{x} \int_{0}^{3} \frac{dx}{\sqrt{x+1}} - \frac{1}{x} \int_{0}^{3} \frac{dx}{\sqrt{x+1}} \\
= \frac{1}{3} \cdot \frac{2}{3} \left[ (x+1)^{3/2} \right]_{0}^{3} - 1 = \frac{4}{3} \\
(We used FORMULA 11 with a = 1, b = 1, n = 1 and a = 1, b = 1, n = -1) \\
\frac{y}{x} = \frac{1}{3} \int_{0}^{3} \frac{dx}{x+1} = \frac{1}{4} \left[ \ln (x+1) \right]_{0}^{3} = \frac{1}{4} \ln 4 = \frac{1}{2} \ln 2 = \ln \sqrt{2} \]

60. \( M_{x} = \int_{0}^{3} x \left( \frac{3x}{5x+3} \right) dx = 18 \int_{0}^{3} \frac{2x+3}{5x+3} dx - 54 \int_{0}^{3} \frac{dx}{5x+3} = \left[ 18x - 27 \ln |2x + 3| \right]_{0}^{3} \)
\[
= 18 \cdot 3 - 27 \ln 9 - (-27 \ln 3) = 54 - 27 \cdot 2 \ln 3 + 27 \ln 3 = 54 - 27 \ln 3 \\
\]

61. \( S = 2\pi \int_{a}^{b} x^2 \sqrt{1 + 4x^2} dx; \)
\[
\begin{align*}
\left[ u = 2x \quad \frac{du}{2} = 2 dx \right] & \rightarrow \left[ u^2 \sqrt{1 + u^2} \right]_{-2}^{2} \\
& = \frac{1}{2} \left[ \frac{9}{2} (1 + 2u^2) \sqrt{1 + u^2} - \frac{1}{8} \ln (u + \sqrt{1 + u^2}) \right]_{-2}^{2} \\
(We used FORMULA 22 with a = 1) & = \frac{1}{2} \left[ \frac{9}{2} (1 + 2 \cdot 4) \sqrt{1 + 4} - \frac{1}{8} \ln (2 + \sqrt{1 + 4}) \right. \\
+ \frac{9}{2} (1 + 2 \cdot 4) \sqrt{1 + 4} + \frac{1}{8} \ln (-2 + \sqrt{1 + 4}) \\
= \frac{9}{8} \sqrt{5} - \frac{1}{8} \ln \left( \frac{2 + \sqrt{5}}{-2 + \sqrt{5}} \right) \approx 7.62 \\
\end{align*} \]

62. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where \( -r < y < -r + d \). The width of this layer is \( 2\sqrt{r^2 - y^2} \). Therefore, \( A = 2 \int_{-r}^{r-d} \sqrt{r^2 - y^2} dy \)
and \( V = L \cdot A = 2L \int_{-r}^{r-d} \sqrt{r^2 - y^2} dy \)

(b) \( 2L \int_{-r}^{r-d} \sqrt{r^2 - y^2} dy = 2L \left[ \frac{1}{2} \sqrt{r^2 - y^2} + \frac{y}{2} \sin^{-1} \left( \frac{y}{r} \right) \right]_{-r}^{r-d} \)
(We used FORMULA 29 with a = r)
\[
= 2L \left[ \frac{1}{2} \sqrt{r^2 - d^2} + \frac{d}{2} \sin^{-1} \left( \frac{d}{r} \right) \right] = 2L \left[ \frac{1}{2} \sqrt{r^2 - d^2} + \left( \frac{d}{2} \right) \sin^{-1} \left( \frac{d}{r} \right) \right] \\
\]

63. The integrand \( f(x) = \sqrt{x - x^2} \) is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from \( x = 0 \) to \( x = 1 \)
\[
\Rightarrow \int_0^1 \sqrt{x - x^2} \, dx = \int_0^1 \sqrt{2 \cdot \frac{1}{2} x - x^2} \, dx = \left[ \frac{(x - \frac{1}{2})}{\frac{1}{2}} \sqrt{2 \cdot \frac{1}{2} x - x^2} + \left( \frac{1}{2} \right)^2 \sin^{-1} \left( \frac{x - \frac{1}{2}}{\frac{1}{2}} \right) \right]_0^1
\]

(We used FORMULA 48 with \( a = \frac{1}{2} \))

\[
= \left[ \frac{(x - \frac{1}{2})}{\frac{1}{2}} \sqrt{x - x^2} + \frac{1}{8} \sin^{-1} (2x - 1) \right]_0^1 = \frac{1}{8} \cdot \frac{\pi}{2} - \frac{1}{8} \left( - \frac{\pi}{2} \right) = \frac{\pi}{8}
\]

64. The integrand is maximized by integrating \( g(x) = x \sqrt{2x - x^2} \) over the largest domain on which \( g \) is nonnegative, namely \([0, 2]\)

\[
\Rightarrow \int_0^2 x \sqrt{2x - x^2} \, dx = \left[ \frac{(x + 1)(2x - 3)\sqrt{2x - x^2}}{6} + \frac{1}{2} \sin^{-1} (x - 1) \right]_0^2
\]

(We used FORMULA 51 with \( a = 1 \))

\[
= \frac{1}{6} \cdot \frac{\pi}{2} - \frac{1}{6} \left( - \frac{\pi}{2} \right) = \frac{\pi}{12}
\]

**CAS EXPLORATIONS**

65. Example CAS commands:

**Maple:**

\[
q1 := \text{Int}(x * \ln(x), x); \quad \# (a)
q1 = \text{value}(q1);
q2 := \text{Int}(x^2 * \ln(x), x); \quad \# (b)
q2 = \text{value}(q2);
q3 := \text{Int}(x^3 * \ln(x), x); \quad \# (c)
q3 = \text{value}(q3);
q4 := \text{Int}(x^4 * \ln(x), x); \quad \# (d)
q4 = \text{value}(q4);
q5 := \text{Int}(x^n * \ln(x), x); \quad \# (e)
q5 = \text{value}(q5);
q6 := \text{simplify}(q6) \text{ assuming } n::\text{integer};
q7 := \text{collect} (\text{factor}(q7), \ln(x));
\]

66. Example CAS commands:

**Maple:**

\[
q1 := \text{Int}(\ln(x)/x, x); \quad \# (a)
q1 = \text{value}(q1);
q2 := \text{Int}(\ln(x)/x^2, x); \quad \# (b)
q2 = \text{value}(q2);
q3 := \text{Int}(\ln(x)/x^3, x); \quad \# (c)
q3 = \text{value}(q3);
q4 := \text{Int}(\ln(x)/x^4, x); \quad \# (d)
q4 = \text{value}(q4);
q5 := \text{Int}(\ln(x)/x^n, x); \quad \# (e)
q5 = \text{value}(q5);
q7 := \text{simplify}(q6) \text{ assuming } n::\text{integer};
q7 := \text{collect} (\text{factor}(q7), \ln(x));
\]
67. Example CAS commands:

Maple:
q := Int( sin(x)^n/(sin(x)^n+cos(x)^n), x=0..Pi/2 );       # (a)
q = value( q );
q1 := eval( q, n=1 );                                      # (b)
for N in [1,2,3,5,7] do
go := eval( q, n=N );
print( q1 = evalf(q1) );
end do:
qq1 := PDEtools[dchange]( x=Pi/2-u, q, [u] );              # (c)
qq2 := subs( u=x, qq1 );
qq3 := q + q = q + qq2;
qq4 := combine( qq3 );
qq5 := value( qq4 );
simplify( qq5/2 );

65-67. Example CAS commands:

Mathematica: (functions may vary)
In Mathematica, the natural log is denoted by Log rather than Ln, Log base 10 is Log[x,10]
Mathematica does not include an arbitrary constant when computing an indefinite integral,
Clear[ f, n ]
f[x_]:=Log[x] / x
Integrate[f[x], x]

For exercise 67, Mathematica cannot evaluate the integral with arbitrary n. It does evaluate the integral (value is /4 in each case) for small values of n, but for large values of n, it identifies this integral as Indeterminate

65. (e) \( \int x^n \ln x \, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n \, dx, \quad \text{n} \neq -1 \)
(We used FORMULA 110 with a = 1, m = 1)
\[
= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} + C = \frac{x^{n+1}}{n+1} \left( \ln x - \frac{1}{n+1} \right) + C
\]

66. (e) \( \int x^{-n} \ln x \, dx = \frac{x^{-n+1}}{1-n} \ln x - \frac{1}{1-n} \int x^{-n} \, dx, \quad \text{n} \neq 1 \)
(We used FORMULA 110 with a = 1, m = 1, n = -n)
\[
= \frac{x^{-n+1}}{1-n} \ln x - \frac{1}{1-n} \left( \frac{x^{-n}}{1-n} \right) + C = \frac{x^{-n+1}}{1-n} \left( \ln x - \frac{1}{1-n} \right) + C
\]

67. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n.
(b) MAPLE and MATHEMATICA get stuck at about n = 5.
(c) Let \( x = \frac{\pi}{2} - u \Rightarrow dx = -du; \quad x = 0 \Rightarrow u = \frac{\pi}{2}, \quad x = \frac{\pi}{2} \Rightarrow u = 0; \)
\[
I = \int_{0}^{\pi/2} \frac{\sin^3 x \, dx}{\sin^x \cos^x} = \int_{0}^{\pi/2} \frac{-\sin^3(\frac{\pi}{2} - u) \, du}{\sin(\frac{\pi}{2} - u) \cos(\frac{\pi}{2} - u)} = \int_{0}^{\pi/2} \frac{\cos^3 u \, du}{\cos u \sin u} = \int_{0}^{\pi/2} \frac{\cos^3 x \, dx}{\cos x \sin x}
\]
\Rightarrow I + I = \int_{0}^{\pi/2} \left( \frac{\sin^3 x + \cos^3 x}{\sin^x \cos^x} \right) \, dx = \int_{0}^{\pi/2} \, dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}
### 8.6 NUMERICAL INTEGRATION

1. \( \int_1^2 x \, dx \)
   
   **I.** (a) For \( n = 4, \Delta x = \frac{b - a}{n} = \frac{2 - 1}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{1}{8} ; \)
   
   \[ \sum \text{mf}(x_i) = 12 \Rightarrow T = \frac{1}{4} (12) = \frac{3}{2} ; \]
   
   \( f(x) = x \Rightarrow f'(x) = 1 \Rightarrow f''(x) = 0 \Rightarrow M = 0 \)

   \( \Rightarrow |E_r| = 0 \)

   (b) \( \int_1^2 x \, dx = \left[ \frac{x^2}{2} \right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow |E_r| = \int_1^2 x \, dx - T = 0 \)

   (c) \( \frac{|E_r|}{\text{True Value}} \times 100 \times 0\% \)

   **II.** (a) For \( n = 4, \Delta x = \frac{b - a}{n} = \frac{2 - 1}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{1}{8} ; \)
   
   \[ \sum \text{mf}(x_i) = 18 \Rightarrow S = \frac{1}{12} (18) = \frac{3}{2} ; \]
   
   \( f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_s| = 0 \)

   (b) \( \int_1^2 x \, dx = \frac{3}{2} \Rightarrow |E_s| = \int_1^2 x \, dx - S = \frac{3}{2} - \frac{3}{2} = 0 \)

   (c) \( \frac{|E_s|}{\text{True Value}} \times 100 \times 0\% \)

2. \( \int_1^3 (2x - 1) \, dx \)
   
   **I.** (a) For \( n = 4, \Delta x = \frac{b - a}{n} = \frac{3 - 1}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{1}{8} ; \)
   
   \[ \sum \text{mf}(x_i) = 24 \Rightarrow T = \frac{1}{4} (24) = 6 ; \]
   
   \( f(x) = 2x - 1 \Rightarrow f'(x) = 2 \Rightarrow f''(x) = 0 \Rightarrow M = 0 \)

   \( \Rightarrow |E_r| = 0 \)

   (b) \( \int_1^3 (2x - 1) \, dx = [x^2 - x]^3_1 = (9 - 3) - (1 - 1) = 6 \Rightarrow |E_r| = \int_1^3 (2x - 1) \, dx - T = 6 - 6 = 0 \)

   (c) \( \frac{|E_r|}{\text{True Value}} \times 100 \times 0\% \)

   **II.** (a) For \( n = 4, \Delta x = \frac{b - a}{n} = \frac{3 - 1}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{1}{8} ; \)
   
   \[ \sum \text{mf}(x_i) = 36 \Rightarrow S = \frac{1}{6} (36) = 6 ; \]
   
   \( f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_s| = 0 \)

   (b) \( \int_1^3 (2x - 1) \, dx = 6 \Rightarrow |E_s| = \int_1^3 (2x - 1) \, dx - S = 6 - 6 = 0 \)

   (c) \( \frac{|E_s|}{\text{True Value}} \times 100 \times 0\% \)

3. \( \int_{-1}^1 (x^2 + 1) \, dx \)
   
   **I.** (a) For \( n = 4, \Delta x = \frac{b - a}{n} = \frac{1 - (-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \Delta x = \frac{1}{4} ; \)
   
   \[ \sum \text{mf}(x_i) = 11 \Rightarrow T = \frac{1}{4} (11) = 2.75 ; \]
   
   \( f(x) = x^2 + 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 \Rightarrow M = 2 \)

   \( \Rightarrow |E_r| \leq \frac{1}{12} \left( \frac{1}{2} \right)^2 (2) = \frac{1}{12} \text{ or } 0.08333 \)

   (b) \( \int_{-1}^1 (x^2 + 1) \, dx = \left[ \frac{x^3}{3} + x \right]_{-1}^1 = \left( \frac{1}{3} + 1 \right) - \left( -\frac{1}{3} - 1 \right) = \frac{8}{3} \Rightarrow E_r = \int_{-1}^1 (x^2 + 1) \, dx - T = \frac{8}{3} - \frac{11}{4} = -\frac{1}{12} \)

   \( \Rightarrow |E_r| = \left| -\frac{1}{12} \right| \approx 0.08333 \)

   (c) \( \frac{|E_r|}{\text{True Value}} \times 100 \approx 3\% \)

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II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} \Rightarrow \Delta x = \frac{1}{2}$;

\[
\sum mf(x_i) = 16 \Rightarrow S = \frac{1}{6} (16) = \frac{8}{3} = 2.6667;
\]

$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_4| = 0

(b) $\int_{-1}^{1} (x^2 + 1) \, dx = \left[ \frac{x^3}{3} + x \right]_{-1}^{1} = \frac{8}{3}$

\[\Rightarrow |E_4| = \int_{-1}^{1} (x^2 + 1) \, dx - S = \frac{8}{3} - \frac{8}{3} = 0\]

(c) $\frac{|E_4|}{\text{True Value}} \times 100 = 0\%

4. $\int_{0}^{a} (x^2 - 1) \, dx$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{0-(2)}{4} = \frac{3}{4} = \frac{1}{2}$

\[
\Rightarrow \Delta x = \frac{1}{2} ; \sum mf(x_i) = 4 \Rightarrow S = \frac{1}{6} (4) = \frac{8}{3} ;
\]

$f(x) = x^2 - 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2$

\[\Rightarrow M = 2 \Rightarrow |E_4| \leq \frac{0.8333}{\frac{1}{6}} (2) = \frac{1}{6} = 0.08333\]

(b) $\int_{-1}^{0} (x^2 - 1) \, dx = \left[ \frac{x^3}{3} - x \right]_{-2}^{0} = 0 - \left( -\frac{8}{3} + 2 \right) = \frac{2}{3} \Rightarrow E_r = \int_{-2}^{0} (x^2 - 1) \, dx - T = \frac{2}{3} - \frac{3}{4} = -\frac{1}{12}$

\[\Rightarrow |E_4| = \frac{1}{12}\]

(c) $\frac{|E_4|}{\text{True Value}} \times 100 = \left( \frac{\frac{1}{12}}{\frac{1}{6}} \right) \times 100 \approx 13\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{0-(2)}{4} = \frac{3}{4} = \frac{1}{2}$

\[\Rightarrow \Delta x = \frac{1}{2} ; \sum mf(x_i) = 4 \Rightarrow S = \frac{1}{6} (4) = \frac{8}{3} ;
\]

$f(x) = x^2 - 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2$

\[\Rightarrow M = 2 \Rightarrow |E_4| \leq \frac{0.8333}{\frac{1}{6}} (2) = \frac{1}{6} = 0.08333\]

(b) $\int_{-1}^{0} (x^2 - 1) \, dx = \frac{2}{3} \Rightarrow |E_4| = \int_{-2}^{0} (x^2 - 1) \, dx - S = \frac{2}{3} - \frac{2}{3} = 0$

(c) $\frac{|E_4|}{\text{True Value}} \times 100 = 0\%$

5. $\int_{0}^{a} (t^3 + t) \, dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} \Rightarrow \Delta x = \frac{1}{2}$

\[
\Rightarrow \Delta x = \frac{1}{2} ; \sum mf(t_i) = 25 \Rightarrow T = \frac{1}{4} (25) = \frac{25}{4} ;
\]

$f(t) = t^3 + t \Rightarrow f'(t) = 3t^2 + 1 \Rightarrow f''(t) = 6t$

\[\Rightarrow M = 12 = f''(2) \Rightarrow |E_4| \leq \frac{2-0}{\frac{12}{4}} (12) = \frac{1}{2}\]

(b) $\int_{0}^{2} (t^3 + t) \, dt = \left[ \frac{t^4}{4} + \frac{t^2}{2} \right]_{0}^{2} = \left( \frac{16}{4} + \frac{4}{2} \right) - 0 = 6 \Rightarrow |E_r| = \int_{0}^{2} (t^3 + t) \, dt - T = 6 - \frac{25}{4} = -\frac{1}{4} \Rightarrow |E_r| = \frac{1}{4}$

(c) $\frac{|E_4|}{\text{True Value}} \times 100 = \left( \frac{\frac{1}{4}}{\frac{1}{4}} \right) \times 100 \approx 4\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} \Rightarrow \Delta x = \frac{1}{2}$

\[
\sum mf(t_i) = 36 \Rightarrow S = \frac{1}{6} (36) = 6 ;
\]

$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_4| = 0

(b) $\int_{0}^{2} (t^3 + t) \, dt = 6 \Rightarrow |E_4| = \int_{0}^{2} (t^3 + t) \, dt - S = 6 - 6 = 0$

(c) $\frac{|E_4|}{\text{True Value}} \times 100 = 0\%$
6. \[ \int_{-1}^{1} (t^3 + 1) \, dt \]

I. (a) For \( n = 4 \), \( \Delta x = \frac{1 - (-1)}{4} = \frac{2}{4} = \frac{1}{2} \):
\[ \sum \text{mf}(t) = 8 \Rightarrow T = \frac{1}{4} (8) = 2 ; \]
\[ f(t) = t^3 + 1 \Rightarrow f'(t) = 3t^2 \Rightarrow f''(t) = 6t \]
\[ M = 6 = f''(1) \Rightarrow |E_r| = \frac{1}{12} \frac{1}{2} (6) = \frac{1}{4} \]
\[ \int_{-1}^{1} (t^3 + 1) \, dt = \left[ \frac{t^4}{4} + t \right]_{-1}^{1} = \left( \frac{1}{4} + 1 \right) - \left( -\frac{1}{4} \right) = 2 \Rightarrow |E_r| = \int_{-1}^{1} (t^3 + 1) \, dt - T = 2 - 2 = 0 \]

(b) \[ \int_{-1}^{1} (t^3 + 1) \, dt = \left[ \frac{t^4}{4} + t \right]_{-1}^{1} = \left( \frac{1}{4} + 1 \right) - \left( -\frac{1}{4} \right) = 2 \Rightarrow |E_r| = \int_{-1}^{1} (t^3 + 1) \, dt - T = 2 - 2 = 0 \]

(c) \[ \frac{|E_r|}{\text{True Value}} \times 100 = 0\% \]

II. (a) For \( n = 4 \), \( \Delta x = \frac{1 - (-1)}{4} = \frac{2}{4} = \frac{1}{2} \):
\[ \sum \text{mf}(t) = 12 \Rightarrow S = \frac{1}{6} (12) = 2 ; \]
\[ f(3)(t) = 6 \Rightarrow f'(4)(t) = 0 \Rightarrow M = 0 \Rightarrow |E_r| = 0 \]
\[ \int_{-1}^{1} (t^3 + 1) \, dt = 2 \Rightarrow |E_r| = \int_{-1}^{1} (t^3 + 1) \, dt - S = 2 - 2 = 0 \]

(c) \[ \frac{|E_r|}{\text{True Value}} \times 100 = 0\% \]

7. \[ \int_{1}^{2} \frac{1}{x} \, dx \]

I. (a) For \( n = 4 \), \( \Delta x = \frac{2 - 1}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{1}{4} ; \)
\[ \sum \text{mf}(s) = \frac{179,573}{44,100} \Rightarrow T = \frac{1}{8} \frac{179,573}{44,100} = \frac{179,573}{352,800} \]
\[ \approx 0.50899; f(s) = \frac{1}{s} \Rightarrow f'(s) = -\frac{2}{s^2} \]
\[ f''(s) = -\frac{3}{s^4} \Rightarrow M = 6 = f''(1) \]
\[ |E_r| \leq \frac{1 - 1}{12} \frac{1}{4} (6) = \frac{1}{32} = 0.03125 \]
\[ \int_{1}^{2} \frac{1}{x} \, dx = \int_{1}^{2} s^{-2} \, ds = \left[ -\frac{1}{2} \right]_{1}^{2} = -\frac{1}{2} - (-\frac{1}{2}) = \frac{1}{2} \Rightarrow |E_r| = \int_{1}^{2} \frac{1}{x} \, dx - T = \frac{1}{2} - 0.50899 = -0.00899 \]
\[ \Rightarrow |E_r| = 0.00899 \]

(c) \[ \frac{|E_r|}{\text{True Value}} \times 100 = \frac{0.00899}{0.5} \times 100 \approx 2\% \]

II. (a) For \( n = 4 \), \( \Delta x = \frac{2 - 1}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{1}{4} ; \)
\[ \sum \text{mf}(s) = \frac{264,821}{44,100} \Rightarrow S = \frac{1}{12} \frac{264,821}{44,100} = \frac{264,821}{329,200} \]
\[ \approx 0.50042; f(3)(s) = -\frac{24}{s^2} \Rightarrow f'(4)(s) = \frac{120}{s^5} \]
\[ f'(s) = -\frac{1}{s^2} \Rightarrow M = 120 \Rightarrow |E_r| \leq \frac{1}{12} \frac{1}{4} (120) \]
\[ = \frac{1}{384} \approx 0.00260 \]
\[ \int_{1}^{2} \frac{1}{x} \, dx = \int_{1}^{2} s^{-2} \, ds = \frac{1}{2} \Rightarrow |E_r| = \int_{1}^{2} \frac{1}{x} \, dx - S = \frac{1}{2} - 0.50042 = -0.00042 \Rightarrow |E_r| = 0.00042 \]

(c) \[ \frac{|E_r|}{\text{True Value}} \times 100 = \frac{0.00042}{0.5} \times 100 \approx 0.08\% \]

8. \[ \int_{2}^{4} \frac{1}{(s-1)^2} \, ds \]

I. (a) For \( n = 4 \), \( \Delta x = \frac{4 - 2}{4} = \frac{1}{2} \Rightarrow \Delta x = \frac{1}{2} ; \)
\[ \sum \text{mf}(s) = \frac{1269}{450} \Rightarrow T = \frac{1}{4} \frac{1269}{450} = \frac{1269}{1800} = 0.70500 ; \]
\[ f(4)(s) = (s-1)^{-2} \Rightarrow f'(4)(s) = -\frac{2}{(s-1)^3} \]
\[ f''(s) = \frac{6}{(s-1)^4} \Rightarrow M = 6 \]
\[ |E_r| \leq \frac{4 - 2}{12} \frac{1}{4} (6) = \frac{1}{4} = 0.25 \]

(c) \[ \frac{|E_r|}{\text{True Value}} \times 100 = 0.25 \]
(b) \[ \int_{1/2}^{4} \frac{1}{x} \, dx = \left[ -\frac{1}{\ln x} \right]_{1/2}^{4} = -\frac{1}{\ln 4} - \left( -\frac{1}{\ln 0.5} \right) = \frac{2}{3} \Rightarrow E_t = \int_{1/2}^{4} \frac{1}{x} \, dx - T = \frac{2}{3} - 0.705 \approx -0.03833 \]
\[ \Rightarrow |E_t| \approx 0.03833 \]
(c) \[ \left( \frac{|E_t|}{\text{True Value}} \right) \times 100 = \left( \frac{0.03833}{0.00481} \right) \times 100 \approx 6\% \]

II. (a) For \( n = 4, \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{1}{8} \):
\[ \sum m f(t_i) = 2 + 2\sqrt{2} \approx 4.8284 \]
\[ \Rightarrow T = \frac{\pi}{8} \left( 2 + 2\sqrt{2} \right) \approx 1.89612; \]
\[ f(t) = \sin t \Rightarrow f'(t) = \cos t \Rightarrow f''(t) = -\sin t \]
\[ \Rightarrow M = 1 \Rightarrow |E_t| \leq \frac{\pi^2}{12} \left( \frac{1}{8} \right)^2 1 = \frac{\pi^2}{192} \approx 0.16149 \]
\[ (b) \int_{0}^{\pi} \sin t \, dt = [\cos t]_0^{\pi} = (\cos \pi) - (\cos 0) = 2 \Rightarrow |E_t| = \int_{0}^{\pi} \sin t \, dt - T \approx 2 - 1.89612 = 0.10388 \]
\[ \left( \frac{|E_t|}{\text{True Value}} \right) \times 100 = \left( \frac{0.10388}{0.00481} \right) \times 100 \approx 5\% \]

II. (a) For \( n = 4, \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{\pi}{8} \):
\[ \sum m f(t_i) = 4 + 2\sqrt{2} \approx 7.6569 \]
\[ \Rightarrow T = \frac{\pi}{8} \left( 4 + 2\sqrt{2} \right) \approx 2.00456; \]
\[ f(t) = -\cos t \Rightarrow f'(t) = \sin t \]
\[ \Rightarrow M = 1 \Rightarrow |E_t| \leq \frac{\pi^2}{12} \left( \frac{\pi}{8} \right)^2 1 = 0.04564 \]
\[ (b) \int_{0}^{\pi} \sin t \, dt = [\cos t]_0^{\pi} = (\cos \pi) - (\cos 0) = 2 \Rightarrow |E_t| = \int_{0}^{\pi} \sin t \, dt - T \approx 2 - 2.00456 = -0.00456 \Rightarrow |E_t| \approx 0.00456 \]
\[ \left( \frac{|E_t|}{\text{True Value}} \right) \times 100 = \left( \frac{0.00456}{0.00481} \right) \times 100 \approx 0\% \]

10. \( \int_{0}^{1} \sin \pi t \, dt \)

I. (a) For \( n = 4, \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{1}{8} \):
\[ \sum m f(t_i) = 2 + 2\sqrt{2} \approx 4.828 \]
\[ \Rightarrow T = \frac{1}{8} \left( 2 + 2\sqrt{2} \right) \approx 0.60355; f(t) = \sin \pi t \]
\[ \Rightarrow f'(t) = \pi \cos \pi t \]
\[ \Rightarrow f''(t) = -\pi^2 \sin \pi t \Rightarrow M = \pi^2 \]
\[ \Rightarrow |E_t| \leq \frac{\pi^2}{12} \left( \frac{1}{4} \right)^2 \left( \frac{\pi^2}{180} \right) \approx 0.05140 \]
\[ (b) \int_{0}^{1} \sin \pi t \, dt = [ -\frac{1}{\pi} \cos \pi t]_0^1 = (-\frac{1}{\pi} \cos \pi) - (-\frac{1}{\pi} \cos 0) = \frac{2}{\pi} \approx 0.63662 \Rightarrow |E_t| = \int_{0}^{1} \sin \pi t \, dt - T \approx \frac{2}{\pi} - 0.60355 = 0.03307 \]
\[ \left( \frac{|E_t|}{\text{True Value}} \right) \times 100 = \left( \frac{0.03307}{\pi} \right) \times 100 \approx 5\% \]

\[ \begin{align*}
\text{Table:} & \quad \begin{array}{cccc}
\text{t_i} & f(t_i) & m & \text{mf(t_i)} \\
0 & 0 & 1 & 0 \\
0.25 & \sqrt{2}/2 & 4 & \sqrt{2} \\
0.5 & 1/2 & 2 & 1/2 \\
0.75 & \sqrt{2}/2 & 4 & \sqrt{2} \\
1 & 0 & 1 & 0 \\
\end{array}
\end{align*} \]
II. (a) For \( n = 4, \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \Delta x = \frac{1}{12} : \)
\[
\sum m f(t_i) = 2 + 4\sqrt{2} \approx 7.65685
\]
\[
\Rightarrow S = \frac{1}{12} \left( 2 + 4\sqrt{2} \right) \approx 0.63807;
\]
\[
f^{(3)}(t) = -\pi^3 \cos \pi t \Rightarrow f^{(4)}(t) = \pi^4 \sin \pi t
\]
\[
\Rightarrow M = \pi^4 \Rightarrow |E_s| \leq \frac{1}{180} \left( \frac{1}{4} \right)^4 (\pi^4) \approx 0.00211
\]
(b) \[ \int_0^1 \sin \pi t \, dt = \frac{2}{\pi} \approx 0.63662 \Rightarrow E_s = \int_0^1 \sin \pi t \, dt - S \approx \frac{2}{\pi} - 0.63807 = -0.00145 \Rightarrow |E_s| \approx 0.00145 \]
(c) \[ \frac{|E_s|}{\text{True Value}} \times 100 = \frac{0.00145}{0.00145} \times 100 = 100 \approx 0\%
\]

11. (a) \( M = 0 \) (see Exercise 1): Then \( n = 1 \Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{1}{12} (1)^2 (0) = 0 < 10^{-4} \)
(b) \( M = 0 \) (see Exercise 1): Then \( n = 2 \) (\( n \) must be even) \( \Rightarrow \Delta x = \frac{1}{2} \Rightarrow |E_s| = \frac{1}{180} \left( \frac{1}{2} \right)^4 (0) = 0 < 10^{-4} \)

12. (a) \( M = 0 \) (see Exercise 2): Then \( n = 1 \Rightarrow \Delta x = 2 \Rightarrow |E_s| = \frac{2}{12} (2)^2 (0) = 0 < 10^{-4} \)
(b) \( M = 0 \) (see Exercise 2): Then \( n = 2 \) (\( n \) must be even) \( \Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180} (1)^4 (0) = 0 < 10^{-4} \)

13. (a) \( M = 2 \) (see Exercise 3): Then \( \Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{12} \left( \frac{2}{n} \right)^2 (2) = \frac{4}{3n} < 10^{-4} \Rightarrow n^2 > \frac{4}{3} (10^4) \Rightarrow n > \sqrt[3]{4} (10^4) \)
\[
\Rightarrow n > 115.4, \text{ so let } n = 116
\]
(b) \( M = 0 \) (see Exercise 3): Then \( n = 2 \) (\( n \) must be even) \( \Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180} (1)^4 (0) = 0 < 10^{-4} \)

14. (a) \( M = 2 \) (see Exercise 4): Then \( \Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{12} \left( \frac{2}{n} \right)^2 (2) = \frac{4}{3n} < 10^{-4} \Rightarrow n^2 > \frac{4}{3} (10^4) \Rightarrow n > \sqrt[3]{4} (10^4) \)
\[
\Rightarrow n > 115.4, \text{ so let } n = 116
\]
(b) \( M = 0 \) (see Exercise 4): Then \( n = 2 \) (\( n \) must be even) \( \Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180} (1)^4 (0) = 0 < 10^{-4} \)

15. (a) \( M = 12 \) (see Exercise 5): Then \( \Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{12} \left( \frac{2}{n} \right)^2 (12) = \frac{8}{3n} < 10^{-4} \Rightarrow n^2 > 8 (10^4) \Rightarrow n > \sqrt[3]{8} (10^4) \)
\[
\Rightarrow n > 282.8, \text{ so let } n = 283
\]
(b) \( M = 0 \) (see Exercise 5): Then \( n = 2 \) (\( n \) must be even) \( \Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180} (1)^4 (0) = 0 < 10^{-4} \)

16. (a) \( M = 6 \) (see Exercise 6): Then \( \Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{12} \left( \frac{2}{n} \right)^2 (6) = \frac{4}{3n} < 10^{-4} \Rightarrow n^2 > 4 (10^4) \Rightarrow n > \sqrt[3]{4} (10^4) \)
\[
= 200, \text{ so let } n = 201
\]
(b) \( M = 0 \) (see Exercise 6): Then \( n = 2 \) (\( n \) must be even) \( \Rightarrow \Delta x = 1 \Rightarrow |E_s| = \frac{2}{180} (1)^4 (0) = 0 < 10^{-4} \)

17. (a) \( M = 6 \) (see Exercise 7): Then \( \Delta x = \frac{1}{n} \Rightarrow |E_s| \leq \frac{1}{12} \left( \frac{1}{n} \right)^2 (6) = \frac{2}{3n} < 10^{-4} \Rightarrow n^2 > \frac{1}{2} (10^4) \Rightarrow n > \sqrt[3]{\frac{1}{2}} (10^4) \)
\[
\Rightarrow n > 70.7, \text{ so let } n = 71
\]
(b) \( M = 120 \) (see Exercise 7): Then \( \Delta x = \frac{1}{n} \Rightarrow |E_s| = \frac{1}{180} \left( \frac{1}{n} \right)^4 (120) = \frac{2}{3n} < 10^{-4} \Rightarrow n^4 > \frac{2}{3} (10^4) \)
\[
\Rightarrow n > \sqrt[3]{\frac{2}{3}} (10^4) \Rightarrow n > 9.04, \text{ so let } n = 10 \) (\( n \) must be even)

18. (a) \( M = 6 \) (see Exercise 8): Then \( \Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{12} \left( \frac{2}{n} \right)^2 (6) = \frac{4}{3n} < 10^{-4} \Rightarrow n^2 > 4 (10^4) \Rightarrow n > \sqrt[3]{4} (10^4) \)
\[
\Rightarrow n > 200, \text{ so let } n = 201
\]
(b) \( M = 120 \) (see Exercise 8): Then \( \Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{180} \left( \frac{2}{n} \right)^4 (120) = \frac{64}{3n} < 10^{-4} \Rightarrow n^4 > \frac{64}{3} (10^4) \)
\[
\Rightarrow n > \sqrt[3]{\frac{64}{3}} (10^4) \Rightarrow n > 21.5, \text{ so let } n = 22 \) (\( n \) must be even)
19. (a) \( f(x) = \sqrt{x + 1} \Rightarrow f'(x) = \frac{1}{2} (x + 1)^{-1/2} \Rightarrow f''(x) = -\frac{1}{4} (x + 1)^{-3/2} = -\frac{1}{4 \sqrt{x + 1}} \Rightarrow M = \frac{1}{4 \sqrt{1}} = \frac{1}{4} \).

Then \( \Delta x = \frac{1}{n} \Rightarrow |E_t| \leq \frac{1}{12} \left( \frac{1}{3} \right)^2 \left( \frac{1}{4} \right) = \frac{1}{72} < 10^{-4} \Rightarrow n^2 > \frac{1}{16} (10^4) \Rightarrow n > \sqrt{\frac{1}{16} (10^4)} \Rightarrow n > 75, \)

so let \( n = 76 \)

(b) \( f^{(3)}(x) = \frac{1}{8} (x + 1)^{-5/2} \Rightarrow f^{(4)}(x) = -\frac{15}{16} (x + 1)^{-7/2} = -\frac{15}{16} \sqrt{x + 1} \Rightarrow M = \frac{15}{16} \sqrt{1} = \frac{15}{16}. \) Then \( \Delta x = \frac{3}{n} \)

\[ |E_s| \leq \frac{3}{180} \left( \frac{3}{n} \right)^4 \left( \frac{15}{16} \right) = \frac{3^3 (15) (10)^4}{16 (180) n^6} < 10^{-4} \Rightarrow n^4 > \frac{3^4 (15) (10)^4}{16 (180) n^6} \Rightarrow n > \frac{\sqrt{3^4 (15) (10)^4}}{16 (180) n^6} \Rightarrow n > 10.6, \]

so let \( n = 12 \) (n must be even)

20. (a) \( f(x) = \frac{1}{\sqrt{x + 1}} \Rightarrow f'(x) = -\frac{1}{2} (x + 1)^{-3/2} \Rightarrow f''(x) = \frac{3}{4} (x + 1)^{-5/2} = \frac{3}{4} \sqrt{x + 1} \Rightarrow M = \frac{3}{4 \sqrt{1}} = \frac{3}{4}. \)

Then \( \Delta x = \frac{2}{n} \Rightarrow |E_t| \leq \frac{3}{12} \left( \frac{1}{2} \right)^2 \left( \frac{3}{4} \right) = \frac{3^3}{48 n^6} < 10^{-4} \Rightarrow n^2 > \frac{3^4 (10)^4}{48} \Rightarrow n > \sqrt{\frac{3^4 (10)^4}{48}} \Rightarrow n > 129.9, \)

so let \( n = 130 \)

(b) \( f^{(3)}(x) = -\frac{15}{16} (x + 1)^{-9/2} \Rightarrow f^{(4)}(x) = \frac{105}{16} (x + 1)^{-7/2} = \frac{105}{16} \sqrt{x + 1} \Rightarrow M = \frac{105}{16} \sqrt{1} = \frac{105}{16}. \)

Then \( \Delta x = \frac{3}{n} \)

\[ |E_s| \leq \frac{3}{180} \left( \frac{3}{n} \right)^4 \left( \frac{105}{16} \right) = \frac{3^3 (105) (10)^4}{16 (180) n^6} < 10^{-4} \Rightarrow n^4 > \frac{3^4 (105) (10)^4}{16 (180) n^6} \Rightarrow n > \frac{\sqrt{3^4 (105) (10)^4}}{16 (180) n^6} \Rightarrow n > 17.25, \]

so let \( n = 18 \) (n must be even)

21. (a) \( f(x) = \sin (x + 1) \Rightarrow f'(x) = \cos (x + 1) \Rightarrow f''(x) = -\sin (x + 1) \Rightarrow M = 1. \) Then \( \Delta x = \frac{4}{n} \Rightarrow |E_t| \leq \frac{4}{12n^2} \left( \frac{2}{3} \right)^2 (1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8 (10)^4}{12} \Rightarrow n > \sqrt{\frac{8 (10)^4}{12}} \Rightarrow n > 81.6, \)

so let \( n = 82 \)

(b) \( f^{(3)}(x) = -\cos (x + 1) \Rightarrow f^{(4)}(x) = \sin (x + 1) \Rightarrow M = 1. \) Then \( \Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{180} \left( \frac{2}{3} \right)^4 (1) = \frac{32}{180 n^6} < 10^{-4} \Rightarrow n^4 > \frac{32 (10)^4}{180} \Rightarrow n > \sqrt[4]{\frac{32 (10)^4}{180}} \Rightarrow n > 6.49, \)

so let \( n = 8 \) (n must be even)

22. (a) \( f(x) = \cos (x + \pi) \Rightarrow f'(x) = -\sin (x + \pi) \Rightarrow f''(x) = -\cos (x + \pi) \Rightarrow M = 1. \) Then \( \Delta x = \frac{2}{n} \)

\[ |E_t| \leq \frac{2}{12} \left( \frac{2}{3} \right)^2 (1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8 (10)^4}{12} \Rightarrow n > \sqrt{\frac{8 (10)^4}{12}} \Rightarrow n > 81.6, \)

so let \( n = 82 \)

(b) \( f^{(3)}(x) = -\cos (x + \pi) \Rightarrow f^{(4)}(x) = \sin (x + \pi) \Rightarrow M = 1. \) Then \( \Delta x = \frac{2}{n} \Rightarrow |E_s| \leq \frac{2}{180} \left( \frac{2}{3} \right)^4 (1) = \frac{32}{180 n^6} < 10^{-4} \Rightarrow n^4 > \frac{32 (10)^4}{180} \Rightarrow n > \sqrt[4]{\frac{32 (10)^4}{180}} \Rightarrow n > 6.49, \)

so let \( n = 8 \) (n must be even)

23. \( \frac{5}{2} (6.0 + 8.2 + 2(9.1) \ldots + 2(12.7) + 13.0)(30) = 15,990 \text{ ft}^3. \)

24. Use the conversion 30 mph = 44 fps (ft per sec) since time is measured in seconds. The distance traveled as the car accelerates from, say, 40 mph = 58.67 fps to 50 mph = 73.33 fps in \( (4.5 - 3.2) = 1.3 \text{ sec} \) is the area of the trapezoid (see figure) associated with that time interval: \( \frac{1}{2} \text{ (58.67 + 73.33) (1.3) = 85.8 \text{ ft}. \) The total distance traveled by the Ford Mustang Cobra is the sum of all these eleven trapezoids (using \( \Delta t/2 \) and the table below):

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<th>0</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
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<th>80</th>
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<th>100</th>
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<th>120</th>
<th>130</th>
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<tr>
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<td>44</td>
<td>58.67</td>
<td>73.33</td>
<td>88</td>
<td>102.67</td>
<td>117.33</td>
<td>132</td>
<td>146.67</td>
<td>161.33</td>
<td>176</td>
<td>190.67</td>
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<tr>
<td>( t ) (sec)</td>
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<td>2.2</td>
<td>3.2</td>
<td>4.5</td>
<td>5.9</td>
<td>7.8</td>
<td>10.2</td>
<td>12.7</td>
<td>16</td>
<td>20.6</td>
<td>26.2</td>
<td>37.1</td>
</tr>
<tr>
<td>( \Delta t/2 )</td>
<td>0</td>
<td>1.1</td>
<td>0.5</td>
<td>0.65</td>
<td>0.7</td>
<td>0.95</td>
<td>1.2</td>
<td>1.25</td>
<td>1.65</td>
<td>2.3</td>
<td>2.8</td>
<td>5.45</td>
</tr>
</tbody>
</table>
25. Using Simpson's Rule, \( \Delta x = 1 \Rightarrow \Delta x = \frac{1}{3} ; \)
\[
\sum m \Delta y = 33.6 \Rightarrow \text{Cross Section Area} \approx \frac{1}{3} (33.6) \]
= 11.2 ft². Let \( x \) be the length of the tank. Then the
Volume \( V = (\text{Cross Sectional Area}) x = 11.2x. \)
Now 5000 lb of gasoline at 42 lb/ft³
\[
\Rightarrow V = \frac{5000}{42} = 119.05 \text{ ft}^3
\]
\[
\Rightarrow 119.05 = 11.2x \Rightarrow x \approx 10.63 \text{ ft}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
x_i & y_i & m & m y_i \\
\hline
x_0 & 0 & 1.5 & 1.5 \\
x_1 & 1 & 1.6 & 4.6 \\
x_2 & 2 & 1.8 & 3.6 \\
x_3 & 3 & 1.9 & 7.6 \\
x_4 & 4 & 2.0 & 8.4 \\
x_5 & 5 & 2.1 & 2.1 \\
x_6 & 6 & 2.1 & 2.1 \\
\hline
\end{array}
\]

26. \( \frac{24}{x} \{0.019 + 2(0.020) + 2(0.021) + \ldots + 2(0.031) + 0.035]\) = 4.2 L.

27. (a) \( |E| \leq \frac{b-a}{180} (\Delta x)^3 M; n = 4 \Rightarrow \Delta x = \frac{x_0 - x_1}{3} ; \)
\[
|f^{(4)}(x)| \leq 1 \Rightarrow M = 1 \Rightarrow |E| \leq \frac{(x_0 - x_1)^4}{180} (1) \approx 0.00021
\]
(b) \( \Delta x = \frac{x_0 - x_1}{3} \Rightarrow \frac{\Delta x}{3} = \frac{x_0 - x_1}{9} ; \)
\[
\sum m f(x_i) = 10.47208705
\]
\[
\Rightarrow S = \frac{3}{2} (10.47208705) \approx 1.37079
\]

(c) \( \approx (0.00021) \times 100 \approx 0.015\%
\]

28. (a) \( \Delta x = \frac{b-a}{n} = \frac{1-0}{10} = 0.1 \Rightarrow \text{erf}(1) = \frac{2}{\sqrt{\pi}} \left(0.1\right) \left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_9 + y_{10}\right)
\]
\[
\Rightarrow \frac{1}{2} (0.01 + 0.04 + 4e^{-0.01} + 2e^{-0.04} + 4e^{-0.09} + \ldots + 4e^{-0.81} + e^{-1}) \approx 0.843
\]
(b) \( |E| \leq \frac{(x_0 - x_1)^4}{180} (0.1)^4(12) = 6.7 \times 10^{-6}
\]

29. \( T = \frac{\Delta x}{3} (y_0 + 2y_1 + 4y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n) \) where \( \Delta x = \frac{b-a}{n} \) and \( f \) is continuous on \([a, b]. \) So
\[
T = \frac{b-a}{n} \left(y_0 + 2y_1 + 4y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n\right) = \frac{b-a}{n} \left(\frac{f(x_0) + f(x_1)}{2} + \frac{f(x_1) + f(x_2)}{2} + \ldots + \frac{f(x_{n-1}) + f(x_n)}{2}\right).
\]
Since \( f \) is continuous on each interval \([x_{k-1}, x_k], \) and \( f(x_{k-1}) + f(x_k) \) is always between \( f(x_{k-1}) \) and \( f(x_k), \) there is a point \( c_k \) in \([x_{k-1}, x_k] \) with \( f(c_k) = \frac{f(x_{k-1}) + f(x_k)}{2}; \) this is a consequence of the Intermediate Value Theorem. Thus our sum is
\[
\sum_{k=1}^{n} \left(\frac{b-a}{n}\right) f(c_k) \text{ which has the form } \sum_{k=1}^{n} \Delta x_k f(c_k) \text{ with } \Delta x_k = \frac{b-a}{n} \text{ for all } k. \text{ This is a Riemann Sum for } f \text{ on } [a, b].
\]

30. \( S = \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n) \) where \( n \) is even, \( \Delta x = \frac{b-a}{n} \) and \( f \) is continuous on \([a, b]. \) So
\[
S = \frac{b-a}{n} \left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right) = \frac{b-a}{n} \left(\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \ldots + \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right)
\]
\[
\frac{f(x_{k+1}) + f(x_{k+2})}{6} \text{ is the average of the six values of the continuous function on the interval } [x_{2k}, x_{2k+2}], \text{ so it is between the minimum and maximum of } f \text{ on this interval. By the Extreme Value Theorem for continuous functions, } f \text{ takes on its maximum and minimum in this interval, so there are } x_k \text{ and } x_{k+1} \text{ with } x_{2k} \leq x_k, x_k \leq x_{2k+2}, \text{ and } f(x_k) \leq \frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6} \leq f(x_{2k+2}); \text{ By the Intermediate Value Theorem, there is } c_k \text{ in } [x_{2k}, x_{2k+2}] \text{ with } f(c_k) = \frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6}. \text{ So our sum has the form } \sum_{k=1}^{n/2} \Delta x_k f(c_k) \text{ with } \Delta x_k = \frac{b-a}{(n/2)}, \text{ a Riemann sum for } f \text{ on } [a, b].
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31. (a) a = 1, e = \frac{1}{2} \Rightarrow \text{Length} = 4 \int_{0}^{\pi/2} \sqrt{1 - \frac{1}{4} \cos^2 t} \, dt
\begin{align*}
&= 2 \int_{0}^{\pi/2} \sqrt{4 - \cos^2 t} \, dt = \int_{0}^{\pi/2} f(t) \, dt; \text{use the Trapezoid Rule with } n = 10 \Rightarrow \Delta t = \frac{\pi - 0}{10} = \frac{\pi}{10}.
&= \frac{\pi}{10} \cdot \sum_{n=0}^{10} \left( f\left( \frac{n \pi}{10} \right) + f\left( \frac{(n + 1) \pi}{10} \right) \right) \\
&= \frac{\pi}{10} \cdot \sum_{n=0}^{10} \left( \frac{37.3686183}{2} \right) \\
&\approx 5.870
\end{align*}

(b) \left| f''(t) \right| < 1 \Rightarrow M = 1
\Rightarrow \left| E_n \right| \leq \frac{M^2}{2} \left( \Delta t \right)^2 \leq \left( \frac{\pi}{100} \right)^2 1 = 0.0032

32. \Delta x = \frac{\pi - 0}{8} = \frac{\pi}{8} \Rightarrow \Delta x = \frac{\pi}{8}; \sum m f(x_n) = 29.184807792
\Rightarrow S = \frac{\pi}{8}(29.18480779) \approx 3.82082

33. The length of the curve \( y = \sin \left( \frac{3\pi}{20} x \right) \) from 0 to 20 is: \( L = \int_{0}^{20} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)
\begin{align*}
&= \int_{0}^{20} \sqrt{1 + \frac{3\pi}{200} \cos \left( \frac{3\pi}{20} x \right)} \, dx. \text{ Using numerical integration we find } L \approx 21.07 \text{ in}
\end{align*}

34. First, we'll find the length of the cosine curve: \( L = \int_{0}^{25} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)
\begin{align*}
\Rightarrow \left( \frac{dy}{dx} \right)^2 &= \frac{\pi}{4} \sin^2 \left( \frac{3\pi}{20} \right) \Rightarrow L = \int_{0}^{25} \sqrt{1 + \left( \frac{\pi}{4} \sin \left( \frac{3\pi}{20} \right) \right)} \, dx. \text{ Using a numerical integrator we find}
\end{align*}
\( L \approx 73.1848 \text{ ft. Surface area is: } A = \text{length} \cdot \text{width} \approx (73.1848)(300) = 21,955.44 \text{ ft.}
\text{Cost} = 1.75A = (1.75)(21,955.44) = \$38,422.02. \text{ Answers may vary slightly, depending on the numerical integration used.}

35. \( y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow \left( \frac{dy}{dx} \right)^2 = \cos^2 x \Rightarrow S = \int_{0}^{\pi} 2\pi x \sqrt{1 + \cos^2 x} \, dx; \text{ a numerical integration gives}
S \approx 14.4
\)

36. \( y = \frac{x^2}{4} \Rightarrow \frac{dy}{dx} = \frac{x}{2} \Rightarrow \left( \frac{dy}{dx} \right)^2 = \frac{x^2}{4} \Rightarrow S = \int_{0}^{\pi} 2\pi \left( \frac{x^2}{4} \right) \sqrt{1 + \frac{x^2}{4}} \, dx; \text{ a numerical integration gives } S \approx 5.28
\)

37. A calculator or computer numerical integrator yields \( \sin^{-1} 0.6 \approx 0.643501109. \)

38. A calculator or computer numerical integrator yields \( \pi \approx 3.1415929. \)

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<th>( f(x_i) )</th>
<th>( m )</th>
<th>( mf(x_i) )</th>
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8.7 IMPROPER INTEGRALS

1. \[ \int_0^\infty \frac{dx}{x^2 + 1} = \lim_{b \to \infty} \int_0^b \frac{dx}{x^2 + 1} = \lim_{b \to \infty} \left[ \tan^{-1} x \right]_0^b = \lim_{b \to \infty} \left( \tan^{-1} b - \tan^{-1} 0 \right) = \frac{\pi}{2} - 0 = \frac{\pi}{2} \]

2. \[ \int_1^\infty \frac{dx}{x^{1/4} - 1} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^{1/4} - 1} = \lim_{b \to \infty} \left[ -1000x^{-0.001} \right]_1^b = \lim_{b \to \infty} \left( -\frac{1000}{b^{0.001}} + 1000 \right) = 1000 \]

3. \[ \int_0^1 \frac{dx}{x} = \lim_{b \to 0^+} \int_0^b x^{-1/2} \, dx = \lim_{b \to 0^+} \left[ 2x^{1/2} \right]_1^b = \lim_{b \to 0^+} \left( 2 - 2\sqrt{b} \right) = 2 - 0 = 2 \]

4. \[ \int_0^1 \frac{dx}{\sqrt{4 - x}} = \lim_{b \to 1^-} \int_0^b (4 - x)^{-1/2} \, dx = \lim_{b \to 1^-} \left[ -2\sqrt{4 - b} - (-2\sqrt{4}) \right] = 0 + 4 = 4 \]

5. \[ \int_0^1 \frac{dx}{x^{1/3} - 1} = \int_0^1 \frac{dx}{x^{1/3} + 1} + \int_0^1 \frac{dx}{x^{1/3} - 1} = \lim_{b \to 0^+} \left[ 3x^{1/3} \right]_0^b + \lim_{c \to 0^+} \left[ 3x^{1/3} \right]_c^1 = \lim_{b \to 0^+} \left[ 3b^{1/3} - 3(-1)^{1/3} \right] + \lim_{c \to 0^+} \left[ 3(1)^{1/3} - 3c^{1/3} \right] = (0 + 3) + (3 - 0) = 6 \]

6. \[ \int_0^1 \frac{dx}{x^{2/3} - 1} = \int_0^1 \frac{dx}{x^{2/3} + 1} + \int_0^1 \frac{dx}{x^{2/3} - 1} = \lim_{b \to 0^+} \left[ \frac{3}{2} x^{2/3} \right]_b^1 + \lim_{c \to 0^+} \left[ \frac{3}{2} x^{2/3} \right]_c^1 = \lim_{b \to 0^+} \left[ \frac{3}{2} b^{2/3} - \frac{3}{2} (-8)^{2/3} \right] + \lim_{c \to 0^+} \left[ \frac{3}{2} (1)^{2/3} - \frac{3}{2} c^{2/3} \right] = \left[ 0 - \frac{3}{2} (4) \right] + \left( \frac{3}{2} - 0 \right) = -\frac{9}{2} \]

7. \[ \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \lim_{b \to 1^-} \left[ \sin^{-1} x \right]_0^b = \lim_{b \to 1^-} \left( \sin^{-1} b - \sin^{-1} 0 \right) = \frac{\pi}{2} - 0 = \frac{\pi}{2} \]

8. \[ \int_0^1 \frac{dx}{x^{1/4} + 1} = \lim_{b \to 0^+} \left[ 1000x^{0.001} \right]_0^b = \lim_{b \to 0^+} \left( 1000 - 1000b^{0.001} \right) = 1000 - 0 = 1000 \]

9. \[ \int_{-\infty}^\infty \frac{2 \, dx}{x^2 + 1} = \int_{-\infty}^\infty \frac{dx}{x^2 + 1} - \int_{-\infty}^0 \frac{dx}{x^2 + 1} = \lim_{b \to -\infty} \left[ \ln |x| + 1 \right]_b^1 - \lim_{b \to -\infty} \left[ \ln |x + 1| \right]_b^1 = \lim_{b \to -\infty} \left[ \ln \left| \frac{x + 1}{x + 1} \right| \right]_b^1 = \lim_{b \to -\infty} \left[ \frac{\ln |x + 1|}{x + 1} \right]_b^1 = \ln 3 - \ln 1 = \ln 3 \]

10. \[ \int_{-\infty}^\infty \frac{2 \, dx}{x^2 + 1} = \int_{-\infty}^\infty \frac{dx}{x^2 + 1} = \lim_{b \to -\infty} \left[ \tan^{-1} \frac{x}{2} \right]_b^1 = \lim_{b \to -\infty} \left( \tan^{-1} 1 - \tan^{-1} \frac{b}{2} \right) = \frac{\pi}{4} - \left( -\frac{\pi}{2} \right) = 3\pi \]

11. \[ \int_0^\infty \frac{2 \, dv}{v^2 + 1} = \lim_{b \to \infty} \left[ 2 \ln \left| \frac{1 - v}{1 + v} \right| \right]_2^b = \lim_{b \to \infty} \left( 2 \ln \left| \frac{b - 1}{b + 1} \right| - 2 \ln \left| \frac{2 - 1}{2} \right| \right) = 2 \ln (1) - 2 \ln \left( \frac{1}{2} \right) = 0 + 2 \ln 2 = \ln 4 \]

12. \[ \int_0^\infty \frac{2 \, dv}{v^2 + 1} = \lim_{b \to \infty} \left[ \ln \left| \frac{1 - v}{1 + v} \right| \right]_2^b = \lim_{b \to \infty} \left( \ln \left| \frac{b - 1}{b + 1} \right| - \ln \left| \frac{2 - 1}{2} \right| \right) = \ln (1) - \ln \left( \frac{1}{2} \right) = 0 + \ln 3 = \ln 3 \]

13. \[ \int_{-\infty}^\infty \frac{2 \, dx}{(x^2 + 1)^2} = \int_0^\infty \frac{2 \, dx}{(x^2 + 1)^2} + \int_0^\infty \frac{2 \, dx}{(x^2 + 1)^2} = \frac{2}{x^2 + 1} \left[ u = x^2 + 1 \right] - 2 \ln \left| \frac{x + 1}{x - 1} \right| = \lim_{b \to \infty} \left[ -\frac{1}{u} \right]_b^1 + \lim_{c \to \infty} \left[ -\frac{1}{u} \right]_c^1 = \lim_{b \to \infty} \left( -1 + \frac{1}{b} \right) + \lim_{c \to \infty} \left[ -\frac{1}{c} \right] = -1 + 0 + 0 + 1 = 0 \]

14. \[ \int_{-\infty}^\infty \frac{2 \, dx}{(x^2 + 4)^2} = \int_{-\infty}^0 \frac{2 \, dx}{(x^2 + 4)^2} + \int_0^\infty \frac{2 \, dx}{(x^2 + 4)^2} = \frac{2}{x^2 + 4} \left[ u = x^2 + 4 \right] - 2 \ln \left| \frac{x + 4}{x - 4} \right| = \int_0^\infty \frac{du}{2u^2} + \int_0^\infty \frac{du}{2u^2} = \frac{1}{\sqrt{2}} \left[ -\frac{1}{u} \right]_b^c + \lim_{c \to \infty} \left( -\frac{1}{\sqrt{2}} \right) = \left( -\frac{1}{2} + 0 \right) + (0 + \frac{1}{2}) = 0 \]
15. \( \int_0^1 \frac{\theta + 1}{\sqrt{\theta + 2 \theta}} \, d\theta \left[ u = \theta^2 + 2\theta \right] \rightarrow \int_0^1 \frac{du}{2\sqrt{u}} = \lim_{b \to 0^+} \int_b^3 \frac{du}{2\sqrt{u}} = \lim_{b \to 0^+} \left[ \sqrt{u} \right]_b^3 = 3 - 0 = \sqrt{3} \)

16. \( \int_0^2 \frac{x + \frac{1}{x}}{\sqrt{4 - x^2}} \, dx = \frac{1}{2} \int_0^2 \frac{2dx}{\sqrt{4 - x^2}} + \frac{1}{2} \int_0^2 \frac{du}{\sqrt{4 - u^2}} \left[ u = 4 - s^2 \right] \rightarrow -\frac{1}{2} \int_0^2 \frac{du}{\sqrt{4 - u^2}} + \lim_{b \to 0^+} \left[ \sqrt{u} \right]_b^3 = \lim_{b \to 0^+} \left[ \sin^{-1} \frac{1}{2} \right]_b^3 = \lim_{b \to 0^+} \left[ \sin^{-1} \frac{1}{2} \right]_b^3 = 2 - 0 + \left( \frac{\pi}{2} - 0 \right) = \frac{\pi}{2} \)

17. \( \int_0^\infty \frac{dx}{(1 + x)^2} = \int_0^\infty \frac{du}{2\sqrt{u}} + \lim_{b \to \infty} \int_0^b \frac{du}{2\sqrt{u}} = \lim_{b \to \infty} \left[ 2\tan^{-1} u \right]_0^b = \lim_{b \to \infty} \left( 2\tan^{-1} b - 2\tan^{-1} 0 \right) = \left( \frac{\pi}{2} - 2(0) \right) = \pi \)

18. \( \int_0^\infty \frac{dx}{1 + x^2} = \lim_{b \to \infty} \left[ \tan^{-1} x \right]_0^b = \lim_{b \to \infty} \left[ \tan^{-1} b - \tan^{-1} 0 \right] = \lim_{b \to \infty} \left( \frac{\pi}{2} - 0 \right) = \frac{\pi}{2} \)

19. \( \int_0^\infty \frac{dv}{1 + v + v^2 + v^3} = \lim_{b \to \infty} \left[ \tan^{-1} v \right]_0^b = \lim_{b \to \infty} \left[ \tan^{-1} b - \tan^{-1} 0 \right] = \lim_{b \to \infty} \left( \frac{\pi}{2} - 0 \right) = \frac{\pi}{2} \)

20. \( \int_0^\infty \frac{16\tan^{-1} x}{1 + 3x} \, dx = \lim_{b \to \infty} \left[ 8\tan^{-1} x \right]_0^b = \lim_{b \to \infty} \left[ 8\tan^{-1} b - 8\tan^{-1} 0 \right] = 8 \left( \frac{\pi}{2} \right)^2 - 8(0) = 2\pi^2 \)

21. \( \int_0^\infty \theta e^\theta \, d\theta = \lim_{b \to \infty} \left[ e^\theta - e^\theta \right]_0^b = \lim_{b \to \infty} \left[ e^b - e^0 \right] = \lim_{b \to \infty} \left[ e^b - e^0 \right] = 1 - \lim_{b \to \infty} \left( \frac{e^b}{e^0} \right) \quad \text{(L'Hôpital's rule for form)} \)

22. \( \int_0^\infty 2e^{-\theta} \sin \theta \, d\theta = \lim_{b \to \infty} \int_0^b 2e^{-\theta} \sin \theta \, d\theta \)

23. \( \int_0^\infty e^{b\theta} \, dx = \lim_{b \to \infty} \left[ e^{b\theta} \right]_0^1 = \lim_{b \to \infty} \left[ 1 - e^0 \right] = 1 - \left( 1 - e^0 \right) = 1 \)

24. \( \int_0^\infty 2xe^{-x} \, dx = \int_0^\infty 2xe^{-x} \, dx + \int_0^\infty 2xe^{-x} \, dx = \lim_{b \to \infty} \left[ -e^{-x} \right]_0^b + \lim_{c \to \infty} \left[ -e^{-x} \right]_0^c = \lim_{b \to \infty} \left[ -1 - (-e^{-b}) \right] + \lim_{c \to \infty} \left[ -e^{-x} - (-1) \right] = (1 - 0) + (0 + 1) = 0 \)

25. \( \int_0^1 x \ln x \, dx = \lim_{b \to 0^+} \left[ \frac{1}{2} \ln x - \frac{1}{3} x^2 \right]_0^b = \left( \frac{1}{2} \ln 1 - \frac{1}{4} \right) - \lim_{b \to 0^+} \left( \frac{b^2}{2} \ln b - \frac{b^3}{4} \right) = -\frac{1}{4} - \lim_{b \to 0^+} \left( \frac{b^2}{2} \ln b - \frac{b^3}{4} \right) = -\frac{1}{4} \)
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26. \( \int_0^1 (-\ln x) \, dx = \lim_{b \to 0^+} \left[ (1-x \ln x) \right]_b^1 = \left[ 1 - 1 \ln 1 \right] - \lim_{b \to 0^+} [b - b \ln b] = 1 - 0 + \lim_{b \to 0^+} \frac{\ln b}{b} = 1 + \lim_{b \to 0^+} \frac{1}{(\ln b)^2} = 1 - 0 = 1 \)

27. \( \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \lim_{b \to 2^-} \left[ \sin^{-1} \frac{x}{2} \right]_0^b = \lim_{b \to 2^-} \left( \sin^{-1} \frac{b}{2} \right) - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2} \)

28. \( \int_0^1 \frac{4 \, dr}{\sqrt{1-r^2}} = \lim_{b \to 1^-} \left[ 2 \sin^{-1} (r^2) \right]_0^b = \lim_{b \to 1^-} \left[ 2 \sin^{-1} (b^2) \right] - 2 \sin^{-1} 0 = 2 \cdot \frac{\pi}{2} - 0 = \pi \)

29. \( \int_1^2 \frac{1}{\sqrt{r^2 - 1}} \, dr = \lim_{b \to 1^+} [\sec^{-1} \frac{b}{r}] = \sec^{-1} 2 - \lim_{b \to 1^+} \sec^{-1} b = \frac{\pi}{3} - 0 = \frac{\pi}{3} \)

30. \( \int_1^2 \frac{x}{\sqrt{x^2 - 1}} \, dx = \lim_{b \to 2^-} \left[ \frac{1}{2} \sec^{-1} \frac{x}{b} \right]_1^b = \lim_{b \to 2^-} \left( \frac{1}{2} \sec^{-1} \frac{b}{2} \right) - \frac{1}{2} \sec^{-1} (\frac{b}{2}) = \frac{1}{2} (\frac{\pi}{3}) - \frac{1}{2} \cdot 0 = \frac{\pi}{6} \)

31. \( \int_{-1}^1 \frac{x}{\sqrt{x^2 + 1}} \, dx = \frac{1}{2} \int_{-1}^1 \frac{dx}{\sqrt{x^2 + 1}} + \lim_{c \to 0^+} \frac{1}{2} \int_{x=0}^{c=0} \left[ -2 \sqrt{-x} \right]_{x=0}^c + \lim_{c \to 0^+} \frac{1}{2} \int_{x=0}^{c=0} \left[ 2 \sqrt{x} \right]_{x=0}^c \)

32. \( \int_0^2 \frac{dx}{\sqrt{1 - x}} = \int_0^1 \frac{dx}{\sqrt{1 - x}} + \int_1^2 \frac{dx}{\sqrt{1 - x}} = \lim_{b \to 1^+} \left[ -2 \sqrt{1-x} \right]_0^b + \lim_{c \to 0^+} \left[ 2 \sqrt{x-1} \right] \)

33. \( \int_1^\infty \frac{1}{\sqrt{x^2 + 6}} \, dx = \lim_{b \to 1^+} \left[ \ln \left( \frac{\sqrt{x^2 + 6}}{\sqrt{x^2 + 6}} \right) \right]_{x=0}^b = \lim_{b \to 1^+} \left( \ln \left( \frac{\sqrt{b^2 + 6}}{\sqrt{1 + 6}} \right) \right) - \ln \left( \frac{\sqrt{1 + 6}}{\sqrt{1 + 6}} \right) = 0 - \ln \left( \frac{1}{\sqrt{2}} \right) = \ln 2 \)

34. \( \int_0^\infty \frac{dx}{(x+1)(x^2+1)} = \lim_{b \to 0^+} \left[ \frac{1}{2} \ln \left( x+1 \right) - \frac{1}{4} \ln \left( x^2 + 1 \right) + \frac{1}{2} \tan^{-1} x \right]_0^b = \lim_{b \to 0^+} \left[ \frac{1}{2} \ln \left( \frac{x+1}{\sqrt{x^2 + 1}} \right) + \frac{1}{2} \tan^{-1} x \right] \)

35. \( \int_0^{\pi/2} \tan \theta \, d\theta = \lim_{b \to \pi/2^-} \left[ -\ln |\cos \theta| \right]_0^b = \lim_{b \to \pi/2^-} \left[ -\ln |\cos b| \right] + \ln 1 = \lim_{b \to \pi/2^-} \left[ -\ln |\cos b| \right] = +\infty, \text{ the integral diverges} \)

36. \( \int_0^{\pi/2} \cot \theta \, d\theta = \lim_{b \to \pi/2^-} \left[ \ln |\sin \theta| \right]_0^b = \lim_{b \to \pi/2^-} \left[ -\ln |\sin b| \right] - \lim_{b \to 0^+} \left[ \ln |\sin b| \right] = +\infty, \text{ the integral diverges} \)

37. \( \int_0^{\pi/2} \sin \theta \, d\theta = \lim_{b \to \pi/2^-} \left[ -\sin \theta \right]_0^b = \lim_{b \to \pi/2^-} \left[ -\sin b \right] + \sin 1 = \lim_{b \to \pi/2^-} \left[ -\sin b \right] = +\infty, \text{ the integral diverges} \)

38. \( \int_{\pi/2}^{\pi} \frac{d\theta}{\sqrt{x^2 - \theta}} \quad \left[ x = \pi - 2\theta \right] \quad - \int_0^{\pi/2} \frac{d\theta}{\sqrt{x^2 - \theta}} \quad \left[ \theta = \frac{\pi}{2} - \frac{\pi}{2} \right] \quad \frac{d\theta}{\sqrt{x^2 - \theta}} \quad \left[ \frac{d\theta}{\sqrt{x^2 - \theta}} \text{ converges} \right. \)

39. \( \int_0^{\frac{\pi}{2}} \frac{1}{x^2 - \frac{1}{x}} \, dx = \int_{\frac{1}{x}}^{1/\ln x} \frac{e^{y} \, dy}{-y^2} = \int_{\frac{1}{\ln x}}^{\infty} e^{-y} \, dy = \lim_{b \to \infty} \left[ -e^{-y} \right]_{b=0}^{y=\infty} = \lim_{b \to \infty} \left[ -e^{-y} - (-e^{-1/\ln x}) \right] = 0 + e^{-1/\ln x} = e^{-1/\ln x}, \text{ so the integral converges} \)
40. \[ \int_0^\infty \frac{e^y}{\sqrt{x}} \, dx; \quad [y = \sqrt{x}] \quad \rightarrow \quad 2\int_0^1 e^y \, dy = 2 - \frac{2}{e}, \text{ so the integral converges.} \]

41. \[ \int_0^\infty \frac{dt}{\sqrt{t + \sin t}}; \quad \text{Since for } 0 \leq t \leq \pi, 0 \leq \frac{1}{\sqrt{t + \sin t}} \leq \frac{1}{\sqrt{t}} \text{ and } \int_0^\infty \frac{dt}{\sqrt{t}} \text{ converges, then the original integral converges as well by the Direct Comparison Test.} \]

42. \[ \int_0^\infty \frac{dt}{1 - \sin t}; \quad \text{let } f(t) = \frac{1}{1 - \sin t} \text{ and } g(t) = \frac{1}{t}, \text{ then } \lim_{t \to 0^+} f(t) = \lim_{t \to 0^+} g(t) = \infty = \lim_{t \to 0^+} \frac{1}{1 - \sin t} = \lim_{t \to 0^+} \frac{1}{1 - \frac{1}{2}t} = \frac{6}{\sin t} \] \[= \lim_{t \to 0^+} 6 \cos t = 6. \text{ Now, } \int_0^1 \frac{dt}{1 - \sin t} = \lim_{b \to 0^+} \left[ - \frac{1}{2b^2} \right]_0^b = -\frac{1}{2b} \lim_{b \to 0^+} \left[ -\frac{1}{2b^2} \right] = +\infty, \text{ which diverges } \Rightarrow \int_0^1 \frac{dt}{1 - \sin t} \text{ diverges by the Limit Comparison Test.} \]

43. \[ \int_0^1 \frac{dx}{1 - x^2} = \int_0^1 \frac{dx}{1 - x^2} + \int_0^1 \frac{dx}{1 - x^2} \text{ and } \int_0^1 \frac{dx}{1 - x^2} = \lim_{b \to 1^-} \left[ \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| \right]_0^b = \lim_{b \to 1^-} \left[ \frac{1}{2} \ln \left| \frac{1 + b}{1 - b} \right| \right] - 0 = 0, \text{ which diverges } \Rightarrow \int_0^1 \frac{dx}{1 - x} \text{ diverges as well.} \]

44. \[ \int_0^1 \ln |x| \, dx = \int_0^1 \ln (1 - x) \, dx + \int_0^1 \ln x \, dx; \quad \int_0^1 \ln x \, dx = \lim_{b \to 0^+} \ln \left[ 1 - \ln (1 - x)^b \right] = -1 - 0 = -1; \quad \int_0^1 \ln (1 - x) \, dx = -1 \Rightarrow \int_0^1 \ln |x| \, dx = -2 \text{ converges.} \]

45. \[ \int_1^\infty \ln |x| \, dx = \int_1^1 \ln (-x) \, dx + \int_0^1 \ln x \, dx; \quad \int_0^1 \ln x \, dx = \lim_{b \to 0^+} \ln \left[ 1 - \ln (1 - x)^b \right] = \lim_{b \to 0^+} \ln \left[ 1 - \ln (1 - b) \right] = 0 = 0, \text{ which diverges } \Rightarrow \int_1^\infty \frac{dx}{1 + x^2} \text{ diverges as well.} \]

46. \[ \int_1^\infty (\ln |x|) \, dx = \int_1^1 (\ln (-x)) \, dx + \int_0^1 (\ln x) \, dx = \lim_{b \to 0^+} \left[ \frac{e^x}{2} \ln x - \frac{e^x}{2} \right]_0^b = \lim_{b \to 0^+} \left[ \frac{e^x}{2} \ln \left( \frac{1 + x}{1 - x} \right) \right]_0^b = \lim_{b \to 0^+} \left[ \frac{e^x}{2} \ln \left( \frac{1 + b}{1 - b} \right) \right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow \text{ the integral converges (see Exercise 25 for the limit calculations).} \]

47. \[ \int_1^\infty \frac{dx}{1 + x^2}; \quad 0 \leq \frac{1}{1 + x^2} \leq \frac{1}{x^2} \text{ for } 1 \leq x < \infty \text{ and } \int_1^\infty \frac{dx}{1 + x^2} \text{ converges } \Rightarrow \int_1^\infty \frac{dx}{1 + x^2} \text{ converges by the Direct Comparison Test.} \]

48. \[ \int_4^\infty \frac{dx}{\sqrt{x - 1}}; \quad \lim_{x \to \infty} \frac{1}{\sqrt{x - 1}} = \lim_{x \to \infty} \frac{\sqrt{x}}{x - 1} = \frac{1}{1 - 0} = 1 \text{ and } \int_4^\infty \frac{dx}{\sqrt{x}} = \lim_{b \to \infty} \left[ 2\sqrt{x} \right]_4^b = \infty, \text{ which diverges } \Rightarrow \int_4^\infty \frac{dx}{\sqrt{x - 1}} \text{ diverges by the Limit Comparison Test.} \]

49. \[ \int_2^\infty \frac{dv}{\sqrt{v - 1}}; \quad \lim_{v \to \infty} \frac{1}{\sqrt{v - 1}} = \lim_{v \to \infty} \frac{\sqrt{v}}{v - 1} = \frac{1}{1 - 0} = 1 \text{ and } \int_2^\infty \frac{dv}{\sqrt{v}} = \lim_{b \to \infty} \left[ 2\sqrt{v} \right]_2^b = \infty, \text{ which diverges } \Rightarrow \int_2^\infty \frac{dv}{\sqrt{v - 1}} \text{ diverges by the Limit Comparison Test.} \]

50. \[ \int_0^\infty \frac{dt}{1 + t^2}; \quad 0 \leq \frac{1}{1 + t^2} \leq \frac{1}{t^2} \text{ for } 0 < t < \infty \text{ and } \int_0^\infty \frac{dt}{1 + t^2} = \lim_{b \to \infty} [-e^{-t}]_0^b = \lim_{b \to \infty} (-e^{-b} + 1) = 1 \Rightarrow \int_0^\infty \frac{dt}{1 + t^2} \text{ converges } \Rightarrow \int_0^\infty \frac{dt}{1 + t^2} \text{ converges by the Direct Comparison Test.} \]

51. \[ \int_0^1 \frac{dx}{\sqrt{x + 1}} = \int_0^1 \frac{dx}{\sqrt{x + 1}} + \int_0^\infty \frac{dx}{\sqrt{x + 1}} < \int_0^1 \frac{dx}{\sqrt{x + 1}} + \int_0^\infty \frac{dx}{x} \text{ and } \int_0^\infty \frac{dx}{x} = \lim_{b \to \infty} \left[ -\frac{1}{x} \right]_1^b = \lim_{b \to \infty} (-\frac{1}{2b^2} + \frac{1}{2}) = \frac{1}{2} \Rightarrow \int_0^\infty \frac{dx}{\sqrt{x + 1}} \text{ converges by the Direct Comparison Test.} \]
52. \( \int_2^\infty \frac{dx}{\sqrt{x^2 - 1}} \); \( x \lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x \to \infty} \frac{1}{\sqrt{1 - \frac{1}{x^2}}} = 1; \int_2^\infty \frac{1}{x} \, dx = \lim_{b \to \infty} [\ln b]_2^b = \infty, \) which diverges \( \Rightarrow \int_2^\infty \frac{dx}{\sqrt{x^2 - 1}} \) diverges by the Limit Comparison Test.

53. \( \int_1^\infty \frac{\sqrt{x + 1}}{x} \, dx \); \( x \lim_{x \to \infty} \frac{\sqrt{x}}{x + 1} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + \frac{1}{x}}} = 1; \int_1^\infty \frac{\sqrt{x}}{x^2} \, dx = \int_1^\infty \frac{dx}{x^{1/2}} \) converges by the Limit Comparison Test.

54. \( \int_2^\infty \frac{x \, dx}{\sqrt{x^2 - 1}} \); \( x \lim_{x \to \infty} \frac{\sqrt{x^2 - 1}}{x} = \lim_{x \to \infty} \frac{1}{\sqrt{1 - \frac{1}{x^2}}} = 1; \int_2^\infty \frac{x \, dx}{\sqrt{x^2 - 1}} = \int_2^\infty \frac{dx}{\sqrt{x^2 - 1}} = \lim_{b \to \infty} [\ln x]_2^b = \infty, \) which diverges \( \Rightarrow \int_2^\infty \frac{x \, dx}{\sqrt{x^2 - 1}} \) diverges by the Limit Comparison Test.

55. \( \int_1^\infty \frac{2 + \cos x}{x} \, dx \); \( 0 < \frac{1}{x} \leq \frac{2 + \cos x}{x} \) for \( x \geq \pi \) and \( \int_1^\infty \frac{dx}{x} = \lim_{b \to \infty} [\ln x]_1^b = \infty, \) which diverges \( \Rightarrow \int_1^\infty \frac{2 + \cos x}{x} \, dx \) diverges by the Direct Comparison Test.

56. \( \int_1^\infty \frac{\tan x}{x} \, dx \); \( 0 < \frac{1}{x} \leq \frac{\tan x}{x} \) for \( x \geq \pi \) and \( \int_1^\infty \frac{dx}{x} = \lim_{b \to \infty} [\ln x]_1^b = \infty, \) which diverges \( \Rightarrow \int_1^\infty \frac{\tan x}{x} \, dx \) converges by the Direct Comparison Test.

57. \( \int_0^1 \frac{2 \, dt}{1 + t} \); \( \lim_{t \to \infty} \frac{2t}{1 + t} = 1 \) and \( \int_0^1 \frac{2 \, dt}{1 + t} = \lim_{b \to \infty} \frac{2}{b} = \frac{2}{b} + 2 = 2 \) \( \Rightarrow \int_0^1 \frac{2 \, dt}{1 + t} \) converges 

58. \( \int_0^1 \frac{dx}{\ln x} \); \( 0 < \frac{1}{x} \leq \frac{1}{\ln x} \) for \( x > 2 \) and \( \int_0^1 \frac{dx}{x} \) diverges \( \Rightarrow \int_0^1 \frac{dx}{\ln x} \) diverges by the Direct Comparison Test.

59. \( \int_1^\infty \frac{e^x}{x} \, dx \); \( 0 < \frac{1}{x} \leq \frac{e^x}{x} \) for \( x > 1 \) and \( \int_1^\infty \frac{dx}{x} \) diverges \( \Rightarrow \int_1^\infty \frac{e^x \, dx}{x} \) diverges by the Direct Comparison Test.

60. \( \int_0^\infty \ln (\ln x) \, dx \); \( x = e^y \) \( \Rightarrow \int_0^\infty \ln (y) \, e^y \, dy \); \( 0 < \ln y < (\ln y) e^y \) for \( y \geq e \) and \( \int_0^\infty \ln y \, dy = \lim_{b \to \infty} [y \ln y - y]_e^b = \infty, \) which diverges \( \Rightarrow \int_0^\infty \ln (\ln x) \, dx \) diverges by the Direct Comparison Test.

61. \( \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \); \( x \lim_{x \to \infty} \frac{1}{\sqrt{x^2 - 1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 - \frac{1}{x^2}}} = 1; \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} = \int_1^\infty e^{-\frac{v^2}{2}} \, dx \)

62. \( \int_0^\infty \frac{dx}{e^{x^2}} \); \( x \lim_{x \to \infty} \frac{e^{x^2}}{x} = \lim_{x \to \infty} \frac{1}{1 - \frac{1}{x^2}} = 1; \int_1^\infty \frac{dx}{e^{x^2}} \) and \( \int_0^\infty \frac{dx}{e^{x^2}} \) diverges \( \Rightarrow \int_0^\infty \frac{dx}{e^{x^2}} \) diverges by the Limit Comparison Test.

63. \( \int_0^\infty \frac{dx}{\sqrt{x^2 + 1}} \); \( \int_0^\infty \frac{dx}{\sqrt{x^2 + 1}} = \int_0^1 \frac{dx}{\sqrt{x^2 + 1}} + \int_1^\infty \frac{dx}{\sqrt{x^2 + 1}} < \int_0^1 \frac{dx}{\sqrt{x^2 + 1}} + \int_1^\infty \frac{dx}{x^2} \) and \( \int_1^\infty \frac{dx}{x^2} = \lim_{b \to \infty} [\frac{1}{b}]_1^b = \lim_{b \to \infty} (-\frac{1}{b} + 1) = 1 \) \( \Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^2 + 1}} \) diverges by the Direct Comparison Test.

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64. \( \int_{0}^{\infty} \frac{dx}{e^{x} + x} = 2 \int_{0}^{\infty} \frac{dx}{e^{x} + x} ; 0 < \frac{1}{e^{x} + x} < \frac{1}{x} \) for \( x > 0 \); \( \int_{0}^{\infty} \frac{dx}{e^{x} + x} \) converges \( \Rightarrow 2 \int_{0}^{\infty} \frac{dx}{e^{x} + x} \) converges by the Direct Comparison Test.

65. (a) \( \int_{0}^{\infty} \frac{dx}{\sqrt[4]{x}} ; [t = \ln x] \rightarrow \int_{0}^{\infty} \frac{dt}{t^{p}} = \lim_{b \to 0} \left[ \frac{1}{p - 1} t^{1-p} \right]_{b}^{1} = \lim_{b \to 0} \frac{b^{p-1}}{p - 1} + \frac{1}{p - 1} (\ln 2)^{1-p} \)

\( \Rightarrow \) the integral converges for \( p < 1 \) and diverges for \( p \geq 1 \)

(b) \( \int_{0}^{\infty} \frac{dx}{x^{4}x^{p}} ; [t = \ln x] \rightarrow \int_{0}^{\infty} \frac{dt}{t^{p}} \) and this integral is essentially the same as in Exercise 65(a): it converges for \( p > 1 \) and diverges for \( p \leq 1 \)

66. \( \int_{0}^{\infty} e^{-x} dx = \lim_{b \to \infty} [e^{-x}]_{0}^{b} = \lim_{b \to \infty} (e^{-b}) - (e^{-0}) = 0 + 1 = 1 \)

67. \( A = \int_{0}^{\infty} e^{-x} dx = \lim_{b \to \infty} \left[ -e^{-x} \right]_{0}^{b} = \lim_{b \to \infty} (-e^{-b}) - (-e^{-0}) = \lim_{b \to \infty} (\ln 1) = 0 \)

68. \( \mathbf{x} = \frac{1}{A} \int_{0}^{\infty} xe^{-\mathbf{x}} dx = \lim_{b \to \infty} [-xe^{-\mathbf{x}} - e^{-\mathbf{x}}]_{0}^{b} = \lim_{b \to \infty} (-be^{-b} - e^{-b}) - (-0 \cdot e^{-0} - e^{-0}) = 0 + 1 = 1 \)

\( \mathbf{y} = \frac{1}{A} \int_{0}^{\infty} (e^{-\mathbf{x}})^{2} dx = \frac{1}{2} \int_{0}^{\infty} e^{-2\mathbf{x}} dx = \lim_{b \to \infty} \left[ \frac{1}{2} \left[ -\frac{1}{2} e^{-2\mathbf{x}} \right] \right]_{0}^{b} = \lim_{b \to \infty} \frac{1}{2} \left[ \frac{1}{2} e^{-2b} \right] - \frac{1}{2} \left( \frac{1}{2} e^{-2b} \right) = 0 + \frac{1}{4} = \frac{1}{4} \)

69. \( V = \int_{0}^{\infty} 2\pi e^{-\mathbf{x}} dx = 2\pi \int_{0}^{\infty} e^{-\mathbf{x}} dx = 2\pi \lim_{b \to \infty} [-xe^{-\mathbf{x}} - e^{-\mathbf{x}}]_{0}^{b} = 2\pi \left[ \lim_{b \to \infty} (-be^{-b} - e^{-b}) - 1 \right] = 2\pi \frac{1}{2} \)

70. \( V = \int_{0}^{\infty} \pi (e^{-\mathbf{x}})^{2} dx = \pi \int_{0}^{\infty} e^{-2\mathbf{x}} dx = \pi \lim_{b \to \infty} \left[ -\frac{1}{2} e^{-2\mathbf{x}} \right]_{0}^{b} = \pi \lim_{b \to \infty} \left( -\frac{1}{2} e^{-2b} + \frac{1}{2} \right) = \pi \frac{1}{2} \)

71. \( A = \int_{0}^{\infty} (\sec \mathbf{x} - \tan \mathbf{x}) d\mathbf{x} = \lim_{b \to \infty} \left[ \ln |\sec \mathbf{x} + \tan \mathbf{x}| - \ln |\sec \mathbf{x}| \right]_{0}^{b} = \lim_{b \to \infty} \left( \ln 1 + \frac{\tan b}{\sec b} - \ln 1 + 0 \right) \)

\( = \lim_{b \to \infty} \ln |1 + \sin b| = \ln 2 \)

72. (a) \( V = \int_{0}^{\sqrt{2}} \pi \sec^{2} \mathbf{x} d\mathbf{x} - \int_{0}^{\sqrt{2}} \pi \tan^{2} \mathbf{x} d\mathbf{x} = \pi \int_{0}^{\sqrt{2}} (\sec^{2} \mathbf{x} - \tan^{2} \mathbf{x}) d\mathbf{x} = \int_{0}^{\sqrt{2}} \pi [\sec^{2} \mathbf{x} - (\sec^{2} \mathbf{x} - 1)] d\mathbf{x} \)

\( = \pi \int_{0}^{\sqrt{2}} d\mathbf{x} = \frac{\pi}{2} \)

(b) \( S_{outer} = \int_{0}^{\sqrt{2}} 2\pi \sec \mathbf{x} \sqrt{1 + \sec^{2} \mathbf{x} \tan^{2} \mathbf{x}} d\mathbf{x} \geq \int_{0}^{\sqrt{2}} 2\pi \sec \mathbf{x} (\sec \mathbf{x} \tan \mathbf{x}) d\mathbf{x} = \pi b \lim_{b \to \sqrt{2}} [\tan^{2} \mathbf{x}]_{0}^{b} \)

\( = \pi b \lim_{b \to \sqrt{2}} \tan^{2} \mathbf{x} = \infty \Rightarrow S_{outer} \) diverges; \( S_{inner} = \int_{0}^{\sqrt{2}} 2\pi \tan \mathbf{x} \sqrt{1 + \sec^{2} \mathbf{x}} d\mathbf{x} \)

\( \geq \int_{0}^{\sqrt{2}} 2\pi \tan \mathbf{x} \sec^{2} \mathbf{x} d\mathbf{x} = \pi b \lim_{b \to \sqrt{2}} [\tan^{2} \mathbf{x}]_{0}^{b} = \pi b \lim_{b \to \sqrt{2}} \tan^{2} \mathbf{x} = \infty \Rightarrow S_{inner} \) diverges
73. (a) \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \lim_{b \to \infty} \left[ e^{-x^2} \right]_b = \lim_{b \to \infty} \left( \frac{3}{2} e^{-x^2} \right) - \left( \frac{3}{2} e^{-1} \right) = 0 + \frac{3}{2} e^{-x^2} = \frac{3}{2} e^{-x^2} \)

\[ \approx 0.0000441 < 0.000042. \] Since \( e^{-x^2} \leq e^{-3x^2} \) for \( x > 3 \), then \( \int_{3}^{\infty} e^{-x^2} \, dx < 0.000042 \) and therefore

\( \int_{0}^{3} e^{-x^2} \, dx \) can be replaced by \( \int_{0}^{3} e^{-3x^2} \, dx \) without introducing an error greater than 0.000042.

(b) \( \int_{0}^{3} e^{-x^2} \, dx \approx 0.88621 \)

74. (a) \( V = \int_{1}^{\infty} \pi \left( \frac{1}{x} \right)^2 \, dx = \pi \lim_{b \to \infty} \left[ \frac{1}{x} \right]_b = \pi \left[ \lim_{b \to \infty} \left( \frac{1}{b} \right) - \left( \frac{1}{1} \right) \right] = \pi(0 + 1) = \pi \)

(b) When you take the limit to \( \infty \), you are no longer modeling the real world which is finite. The comparison step in the modeling process discussed in Section 4.2 relating the mathematical world to the real world fails to hold.

75. (a) ![Graph of \( \sin(t)/t \)](attachment:graph.png)

(b) \( \int((\sin(t))/t, \, t=0..\infty) \); (answer is \( \frac{\pi}{2} \))

76. (a) ![Graph of \( \text{erf}(x) = \int_{0}^{x} \frac{2e^{-t^2}}{\sqrt{\pi}} \, dt \)](attachment:graph.png)

(b) \( \text{f} := 2e^{-(t^2)/2}/\sqrt{\pi} \);

\( \int(f, \, t=0..\infty) \); (answer is 1)

77. (a) \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \)

f is increasing on \((-\infty, 0]\), f is decreasing on \([0, \infty)\).

f has a local maximum at \((0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)\)
(b) Maple commands:

\[
\begin{align*}
> f &:= \exp(-x^2/2) (\sqrt{2\pi})^n; \\
> \text{Int}(f, x = -1..1); &\approx 0.683 \\
> \text{Int}(f, x = -2..2); &\approx 0.954 \\
> \text{Int}(f, x = -3..3); &\approx 0.997
\end{align*}
\]

(c) Part (b) suggests that as \( n \) increases, the integral approaches 1. We can take \( \int_{-n}^{n} f(x) \, dx \) as close to 1 as we want by choosing \( n > 1 \) large enough. Also, we can make \( \int_{0}^{n} f(x) \, dx \) and \( \int_{-n}^{0} f(x) \, dx \) as small as we want by choosing \( n \) large enough. This is because \( 0 < f(x) < e^{-x^2} \) for \( x > 1 \). (Likewise, \( 0 < f(x) < e^{-x^2} \) for \( x < -1 \).)

Thus, \( \int_{0}^{n} e^{-x^2/2} \, dx < \int_{0}^{n} e^{-x^2} \, dx \).

\[
\int_{0}^{n} e^{-x^2/2} \, dx = \lim_{c \to \infty} \int_{c}^{n} e^{-x^2} \, dx = \lim_{c \to \infty} \left[ -2e^{-x^2/2} \right]_{c}^{n} = \lim_{c \to \infty} \left[ -2e^{-c^2/2} + 2e^{-n^2/2} \right] = 2e^{-n^2/2}
\]

As \( n \to \infty \), \( 2e^{-n^2/2} \to 0 \), for large enough \( n \). \( \int_{-n}^{n} f(x) \, dx \) is as small as we want. Likewise for large enough \( n \), \( \int_{-\infty}^{-n} f(x) \, dx \) is as small as we want.

78. (a) The statement is true since \( \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{n}^{b} f(x) \, dx \), \( \int_{a}^{b} f(x) \, dx = \int_{a}^{\infty} f(x) \, dx - \int_{a}^{n} f(x) \, dx \) and \( \int_{a}^{b} f(x) \, dx \) exists since \( f(x) \) is integrable on every interval \( [a, b] \).

(b) \( \int_{a}^{\infty} f(x) \, dx + \int_{n}^{\infty} f(x) \, dx = \int_{a}^{\infty} f(x) \, dx + \int_{n}^{b} f(x) \, dx + \int_{b}^{\infty} f(x) \, dx + \int_{n}^{\infty} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{n}^{b} f(x) \, dx + \int_{b}^{\infty} f(x) \, dx + \int_{n}^{\infty} f(x) \, dx \)

79. Example CAS commands:

\textbf{Maple:}

\[
\begin{align*}
\text{f} &:= (x,p) -> x^p*ln(x); \\
\text{domain} &:= 0..\exp(1); \\
\text{fn_list} &:= \text{seq} (\text{f}(x,p), p=-2..2 ); \\
\text{plot} (\text{fn_list}, x=\text{domain}, y=-50..10, \text{color}=[\text{red,blue,green,cyan,pink}], \text{linestyle}=[1,3,4,7,9], \text{thickness}=[3,4,1,2,0], \\
\text{legend}=["p=-2","p=-1","p=0","p=1","p=2"], \text{title}="#79 (Section 8.7)" ); \\
\text{q1} &:= \text{Int} (\text{f}(x,p), x=\text{domain} ); \\
\text{q2} &:= \text{value} (\text{q1} ); \\
\text{q3} &:= \text{simplify} (\text{q2} ) \text{ assuming } p>0; \\
\text{q4} &:= \text{simplify} (\text{q2} ) \text{ assuming } p<0; \\
\text{q5} &:= \text{value} (\text{eval} (\text{q1}, p=1 ) ); \\
\text{i1} &:= \text{q1} = \text{piecewise} (\text{p<-1}, \text{q4}, \text{p=-1}, \text{q5}, \text{p>1}, \text{q3} );
\end{align*}
\]

80. Example CAS commands:

\textbf{Maple:}

\[
\begin{align*}
\text{f} &:= (x,p) -> x^p*ln(x); \\
\text{domain} &:= \exp(1)..\infty; \\
\text{fn_list} &:= \text{seq} (\text{f}(x,p), p=-2..2 ); \\
\text{plot} (\text{fn_list}, x=\exp(1)..10, y=0..100, \text{color}=[\text{red,blue,green,cyan,pink}], \text{linestyle}=[1,3,4,7,9], \text{thickness}=[3,4,1,2,0], \\
\text{legend}=["p=-2","p=-1","p=0","p=1","p=2"], \text{title}="#80 (Section 8.7)" ); \\
\text{q6} &:= \text{Int} (\text{f}(x,p), x=\text{domain} ); \\
\text{q7} &:= \text{value} (\text{q6} ); \\
\text{q8} &:= \text{simplify} (\text{q7} ) \text{ assuming } p>0; \\
\text{q9} &:= \text{simplify} (\text{q7} ) \text{ assuming } p<0;
\end{align*}
\]
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q10 := value( eval( q6, p=-1 ) );
i2 := q6 = piecewise( p<-1, q9, p=-1, q10, p>-1, q8 );

81. Example CAS commands:

Maple:
\[
f := (x,p) -> x^p*ln(x);
domain := 0..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=0..10, y=-50..50, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
legend=['"p = -2"","p = -1","p = 0","p = 1","p = 2"], title=">81 (Section 8.7)" );
q11 := Int( f(x,p), x=domain );
q11 = lhs(i1+i2);
`=` = rhs(i1+i2);
`=` = piecewise( p<-1, q4+q9, p=-1, q5+q10, p>-1, q3+q8 );
`=` = piecewise( p<-1, -infinity, p=-1, undefined, p>-1, infinity );

82. Example CAS commands:

Maple:
\[
f := (x,p) -> x^p*ln(abs(x));
domain := -infinity..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=-4..4, y=-20..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
legend=['"p = -2"","p = -1","p = 0","p = 1","p = 2"], title=">82 (Section 8.7)" );
q12 := Int( f(x,p), x=domain );
q12p := Int( f(x,p), x=0..infinity );
q12n := Int( f(x,p), x=-infinity..0 );
q12 = q12p + q12n;
`=` = simplify( q12p+q12n );

79-82. Example CAS commands:

Mathematica: (functions and domains may vary)

\[
Clear[x, f, p]
f[x_] := x^p Log[Abs[x]]
int = Integrate[f[x], {x, e, 100}]
int /. p -> 2.5
\]
In order to plot the function, a value for p must be selected.
\[
p = 3;
Plot[f[x], {x, 2.72, 10}]\]

CHAPTER 8 PRACTICE EXERCISES

1. \( u = \ln(x+1), \ du = \frac{dx}{x+1}; \ dv = dx, \ v = x; \)
   \[
   \int \ln(x+1) \ dx = x \ln(x+1) - \int \frac{x}{x+1} \ dx = x \ln(x+1) - \int dx + \int \frac{dx}{x+1} = x \ln(x+1) - x + \ln(x+1) + C_1
   = (x+1) \ln(x+1) - x + C_1 = (x+1) \ln(x+1) - (x+1) + C, \) where \( C = C_1 + 1 \)

2. \( u = \ln x, \ du = \frac{dx}{x}; \ dv = x^2 \ dx, \ v = \frac{1}{3} x^3; \)
   \[
   \int x^2 \ln x \ dx = \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^3 \ \left( \frac{1}{x} \right) \ dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C
   \]
3. \( u = \tan^{-1} 3x, \ du = \frac{3\ dx}{1+9x^2}; \ dv = dx, \ v = x; \)
\[
\int \tan^{-1} 3x \ dx = x \tan^{-1} 3x - \frac{1}{9} \int \frac{dy}{y^2} \]
\[
= x \tan^{-1} (3x) - \frac{1}{6} \ln (1 + 9x^2) + C\]

4. \( u = \cos^{-1} \left( \frac{x}{2} \right), \ du = \frac{-dx}{\sqrt{4-x^2}}; \ dv = dx, \ v = x; \)
\[
\int \cos^{-1} \left( \frac{x}{2} \right) \ dx = x \cos^{-1} \left( \frac{x}{2} \right) + \frac{1}{2} \int \frac{dy}{\sqrt{4-x^2}} \]
\[
= x \cos^{-1} \left( \frac{x}{2} \right) - \sqrt{4-x^2} + C = x \cos^{-1} \left( \frac{x}{2} \right) - 2\sqrt{1 - \left( \frac{x}{2} \right)^2} + C\]

5. \( e^x \)
\[
\begin{align*}
(x + 1)^2 & \rightarrow e^x \\
2(x + 1) & \rightarrow -e^x \\
2 & \rightarrow e^x \\
0 & \Rightarrow \int (x + 1)^2 e^x \ dx = [(x + 1)^2 - 2(x + 1) + 2] e^x + C
\end{align*}\]

6. \( \sin (1 - x) \)
\[
\begin{align*}
x^2 & \rightarrow \cos (1 - x) \\
2x & \rightarrow -\sin (1 - x) \\
2 & \rightarrow -\cos (1 - x) \\
0 & \Rightarrow \int x^2 \sin (1 - x) \ dx = x^2 \cos (1 - x) + 2x \sin (1 - x) - 2 \cos (1 - x) + C
\end{align*}\]

7. \( u = \cos 2x, \ du = -2 \sin 2x \ dx; \ dv = e^x \ dx, \ v = e^x; \)
\[
I = \int e^x \cos 2x \ dx = e^x \cos 2x + 2 \int e^x \sin 2x \ dx; \]
\[
u = \sin 2x, \ du = 2 \cos 2x \ dx; \ dv = e^x \ dx, \ v = e^x; \]
\[
I = e^x \cos 2x + 2 \left[ e^x \sin 2x - 2 \int e^x \cos 2x \ dx \right] = e^x \cos 2x + 2e^x \sin 2x - 4I \Rightarrow I = \frac{e^x \cos 2x}{2} + \frac{2e^x \sin 2x}{3} + C\]

8. \( u = \sin 3x, \ du = 3 \cos 3x \ dx; \ dv = e^{-2x} \ dx, \ v = -\frac{1}{2} e^{-2x}; \)
\[
I = \int e^{-2x} \sin 3x \ dx = -\frac{1}{2} e^{-2x} \sin 3x + \frac{3}{2} \int e^{-2x} \cos 3x \ dx; \]
\[
u = \cos 3x, \ du = -3 \sin 3x \ dx; \ dv = e^{-2x} \ dx, \ v = -\frac{1}{2} e^{-2x}; \]
\[
I = -\frac{1}{2} e^{-2x} \sin 3x + \frac{3}{2} \left[ -\frac{1}{2} e^{-2x} \cos 3x - \frac{3}{2} \int e^{-2x} \sin 3x \ dx \right] = -\frac{1}{2} e^{-2x} \sin 3x - \frac{3}{4} e^{-2x} \cos 3x - \frac{9}{4} I \]
\[
\Rightarrow I = \frac{4}{13} \left( -\frac{1}{2} e^{-2x} \sin 3x - \frac{3}{4} e^{-2x} \cos 3x \right) + C = -\frac{2}{15} e^{-2x} \sin 3x - \frac{3}{15} e^{-2x} \cos 3x + C\]

9. \( \int \frac{x \ dx}{\sqrt{x^2 - 3x + 2}} = \int \frac{2 \ dx}{x-2} - \int \frac{dx}{x-1} = 2 \ln |x - 2| - \ln |x - 1| + C\]

10. \( \int \frac{x \ dx}{\sqrt{x^2 + 4x + 3}} = \frac{3}{2} \int \frac{dx}{x+3} - \frac{1}{2} \int \frac{dx}{x+1} = \frac{3}{2} \ln |x + 3| - \frac{1}{2} \ln |x + 1| + C\]

11. \( \int \frac{dx}{\sqrt{x^2 + 1}^{3/2}} = \int \left( \frac{1}{x} - \frac{1}{x+1} + \frac{-1}{(x+1)^2} \right) \ dx = \ln |x| - \ln |x + 1| + \frac{1}{2} + C\]

12. \( \int \frac{x+1}{x \sqrt{x(x-1)}} \ dx = \int \left( \frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2} \right) \ dx = 2 \ln \left| \frac{x+1}{x} \right| + \frac{1}{2} + C = -2 \ln |x| + \frac{1}{2} + 2 \ln |x - 1| + C\)
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13. \[ \int \frac{\sin \theta}{\cos \theta + \cos \theta - 2} \; d\theta = \frac{\cos \theta - 1}{\cos \theta + 1} + C \]

14. \[ \int \frac{\cos \theta}{\sin \theta + \sin \theta - 2} \; d\theta = \frac{\cos \theta + 1}{\cos \theta - 1} + C \]

15. \[ \int \frac{3x^2 + 4x + 4}{x^2 + 4} \; dx = 4 \ln|x| - \frac{1}{2} \ln(x^2 + 1) + 4 \tan^{-1} x + C \]

16. \[ \int \frac{4x \; dx}{x^2 + 4x + 1} = 2 \tan^{-1} \left( \frac{x}{2} \right) + C \]

17. \[ \int \frac{(v + 3)}{2v^2 - 8v} \; dv = \frac{3}{8} \ln|v| + \frac{5}{16} \ln|v - 2| + \frac{1}{16} \ln|v + 2| + C \]

18. \[ \int \frac{3v - 7}{(v - 1)(v - 2)} \; dv = \frac{1}{6} \ln|t^2 - 2| - \frac{1}{6} \ln(t^2 + 1) + C \]

19. \[ \int \frac{x^2 + x}{x^2 + x^2} \; dx = \int x \; dx + \frac{3}{2} \int \frac{dx}{x + 2} + \frac{1}{3} \int \frac{dx}{x + 1} = \frac{x^2}{2} + \frac{3}{2} \ln|x + 2| + \frac{1}{3} \ln|x - 1| + C \]

20. \[ \int \frac{2x^2 + x + 3x}{x^2 + x + 2} \; dx = \int \left( 2x - 3 + \frac{2x + 2}{x^2 + x + 2} \right) \; dx = \int 2x \; dx + \frac{3}{2} \int \frac{dx}{x + 1} - \frac{9}{2} \int \frac{dx}{x + 2} = x^2 - 3x + \frac{3}{2} \ln|x + 2| + \frac{3}{2} \ln|x + 1| + C \]

21. \[ \int \frac{dx}{(3\sqrt{x})} = \frac{u}{\sqrt{x}} \int \frac{du}{u} = \frac{1}{3} \ln|u - 1| - \frac{1}{3} \ln|u + 1| + C \]

22. \[ \int \frac{dx}{x^2 + 4x + 1} = \int \frac{dx}{x^2 + 4x + 4 - 3} = \frac{1}{3} \ln|x - 1| - \frac{1}{3} \ln|x + 1| + C \]

23. \[ \int \frac{dx}{x^2 + 4x + 1} = \int \left( 2x - 3 + \frac{2x + 2}{x^2 + x + 2} \right) \; dx = \int 2x \; dx + \frac{3}{2} \int \frac{dx}{x + 1} - \frac{9}{2} \int \frac{dx}{x + 2} = x^2 - 3x + \frac{3}{2} \ln|x + 2| + \frac{3}{2} \ln|x + 1| + C \]

24. \[ \int \frac{2x^2 + x + 3x}{x^2 + x + 2} \; dx = \int \left( 2x - 3 + \frac{2x + 2}{x^2 + x + 2} \right) \; dx = \int 2x \; dx + \frac{3}{2} \int \frac{dx}{x + 1} - \frac{9}{2} \int \frac{dx}{x + 2} = x^2 - 3x + \frac{3}{2} \ln|x + 2| + \frac{3}{2} \ln|x + 1| + C \]

25. \[ \int \frac{dx}{x^3(x^2 + 1)} = \int \frac{u}{\sqrt{u^2 + 1}} \; du = \frac{1}{2} \int \frac{du}{u^2 + 1} = \frac{1}{2} \ln|u + 1| - \frac{1}{2} \ln|u - 1| + C \]

26. \[ \int \frac{dx}{x(1 + \sqrt{x})} = \int \frac{u}{\sqrt{u^2 + 1}} \; du = \frac{1}{2} \int \frac{du}{u^2 + 1} = \frac{1}{2} \ln|u + 1| - \frac{1}{2} \ln|u - 1| + C \]

27. \[ \int \frac{e^x - 1}{e^x} \; dx = -\int \frac{du}{u + 1} + \int \frac{du}{u - 1} = \ln|u + 1| + C = \ln|e^{x-1}| + C = \ln|1 - e^{-x}| + C \]

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28. \[
\int \frac{ds}{\sqrt{e^s + 1}} \quad \left[ u = \sqrt{e^s + 1} \right] \quad \Rightarrow \quad \int \frac{2a du}{u(u^2 - 1)} = 2 \int \frac{du}{u - 1} - \int \frac{du}{u + 1} = \ln \left| \frac{u - 1}{u + 1} \right| + C
\]
29. (a) \[
\int \frac{y dy}{\sqrt{16 - y^2}} = -\frac{1}{2} \int \frac{d(16 - y^2)}{\sqrt{16 - y^2}} = -\sqrt{16 - y^2} + C
\]
(b) \[
\int \frac{\sqrt{x^2}}{\sqrt{16 - y^2}} : [y = 4 \sin x] \quad \Rightarrow \quad 4 \int \frac{\sin x \cos x dx}{\cos x} = -4 \cos x + C = -\frac{4\sqrt{16 - y^2}}{-4} + C = -\sqrt{16 - y^2} + C
\]
30. (a) \[
\int \frac{x dx}{\sqrt{4 + x^2}} = \frac{1}{2} \int \frac{d(4 + x^2)}{\sqrt{4 + x^2}} = \sqrt{4 + x^2} + C
\]
(b) \[
\int \frac{x dx}{\sqrt{4 + x^2}} : [x = 2 \tan y] \quad \Rightarrow \quad \int \frac{2 \tan y \cdot 2 \sec^2 y dy}{2 \sec y} = 2 \int \sec y \tan y dy = 2 \sec y + C = \sqrt{4 + x^2} + C
\]
31. (a) \[
\int \frac{u}{\sqrt{9 - u^2}} \quad \left[ u = 9 - x^2 \right] \quad \Rightarrow \quad -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln |u| + C = \ln \frac{1}{\sqrt{9 - x^2}} + C
\]
(b) \[
\int \frac{dx}{\sqrt{4t^2 - 1}} : [t = \frac{1}{2} \sec \theta] \quad \Rightarrow \quad \frac{1}{4} \int \sec^2 \theta d\theta = \frac{\tan \theta}{4} + C = \frac{\sqrt{4t^2 - 1}}{4} + C
\]
32. (a) \[
\int \frac{t dt}{\sqrt{4t^2 - 1}} = \frac{1}{8} \int \frac{d(4t^2 - 1)}{\sqrt{4t^2 - 1}} = \frac{1}{8} \sqrt{4t^2 - 1} + C
\]
(b) \[
\int \frac{t dt}{\sqrt{4t^2 - 1}} : [t = \frac{1}{2} \sec \theta] \quad \Rightarrow \quad \frac{1}{4} \int \sec^2 \theta d\theta = \frac{\tan \theta}{4} + C = \frac{\sqrt{4t^2 - 1}}{4} + C
\]
33. \[
\int \frac{dx}{\sqrt{9 - x^2}} \quad \left[ u = 9 - x^2 \right] \quad \Rightarrow \quad -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln |u| + C = \ln \frac{1}{\sqrt{9 - x^2}} + C
\]
34. \[
\int \frac{dx}{x(9 - x^2)} = \frac{1}{9} \int \frac{dx}{x} + \frac{1}{18} \int \frac{dx}{3-x} - \frac{1}{18} \int \frac{dx}{x+3} = \frac{1}{9} \ln |x| - \frac{1}{18} \ln |3 - x| - \frac{1}{18} \ln |3 + x| + C
\]
35. \[
\int \frac{dx}{9 - x^2} = \frac{1}{6} \int \frac{dx}{3-x} + \frac{1}{6} \int \frac{dx}{3+x} = -\frac{1}{6} \ln |3 - x| + \frac{1}{6} \ln |3 + x| + C = \frac{1}{6} \ln \left(\frac{x + 3}{x - 3}\right) + C
\]
36. \[
\int \frac{dx}{\sqrt{9 - x^2}} : \left[ \frac{x}{3} = 3 \sin \theta \right] \quad \Rightarrow \quad \int \frac{3 \cos \theta}{3 \cos \theta} d\theta = \int d\theta = \theta + C = \sin^{-1} \frac{x}{3} + C
\]
37. \[
\int \sin^2 x \cos^2 x dx = \int \cos^2 x \sin^2 x dx = \int \cos^4 x dx = \int \frac{\cos^6 x}{3} + \frac{\cos^2 x}{3} + C
\]
38. \[
\int \cos^2 x \sin^2 x dx = \int \sin^2 x \cos^2 x dx = \int \sin^2 x \left(1 - \sin^2 x\right) \cos x dx
\]
\[
= \int \sin^2 x \cos x dx - 2 \int \sin^2 x \cos x dx + \int \sin^6 x dx = \int \frac{\sin^2 x}{6} - \frac{\sin^4 x}{8} + \frac{\sin^6 x}{10} + C
\]
39. \[
\int \tan^4 x \sec^2 x dx = \frac{\tan^6 x}{3} + C
\]
40. \[
\int \tan^3 x \sec^3 x dx = \int \left(\sec^2 x - 1\right) \sec^2 x \cdot \sec x \cdot \tan x dx = \int \sec^4 x \cdot \sec x \cdot \tan x dx - \int \sec^2 x \cdot \sec x \cdot \tan x dx
\]
\[
= \frac{\sec^6 x}{6} - \sec^4 x + C
\]

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41. \[
\int \sin 5\theta \cos 6\theta \, d\theta = \frac{1}{2} \int (\sin(-\theta) + \sin(11\theta)) \, d\theta = \frac{1}{2} \int \sin(-\theta) \, d\theta + \frac{1}{2} \int \sin(11\theta) \, d\theta = \frac{1}{2} \cos(-\theta) - \frac{1}{22} \cos 11\theta + C
\]

42. \[
\int \cos 3\theta \cos 6\theta \, d\theta = \frac{1}{2} \int (\cos 0 + \cos 6\theta) \, d\theta = \frac{1}{2} \int d\theta + \frac{1}{2} \int \cos 6\theta \, d\theta = \frac{1}{2} \theta + \frac{1}{12} \sin 6\theta + C
\]

43. \[
\int \sqrt{1 + \cos(\frac{x}{2})} \, dt = \int \sqrt{2} \cos \frac{1}{2} \, dt = 4\sqrt{2} \sin \frac{1}{2} + C
\]

44. \[
\int e^\sqrt{\tan^2 e^t + 1} \, dt = \int |e^t| \sec e^t \, dt = \ln|\sec e^t + \tan e^t| + C
\]

45. \[
|E_t| \leq \frac{3-1}{180} (\Delta x)^3 M \text{ where } \Delta x = \frac{1-1}{n} = \frac{2}{n}; f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4} \Rightarrow f^{(4)}(x) = 24x^{-5} \text{ which is decreasing on [1, 3] } \Rightarrow \text{ maximum of } f^{(4)}(x) \text{ on [1, 3] is } f^{(4)}(1) = 24 \Rightarrow M = 24. \text{ Then } |E_t| \leq 0.0001 \Rightarrow \left(\frac{3-1}{180}\right) \left(\frac{2}{n}\right)^4 (24) \leq 0.0001 \Rightarrow \left(\frac{768}{180}\right) \left(\frac{1}{n^4}\right) \leq 0.0001 \Rightarrow \frac{1}{n^4} \leq (0.0001) \left(\frac{180}{768}\right) \Rightarrow n^4 \geq 10,000 \left(\frac{768}{180}\right) \Rightarrow n \geq 14.37 \Rightarrow n \geq 16 \text{ (n must be even)}.
\]

46. \[
|E_t| \leq \frac{1-0}{12} (\Delta x)^2 M \text{ where } \Delta x = \frac{1-0}{n} = \frac{1}{n}; 0 \leq f''(x) \leq 8 \Rightarrow M = 8. \text{ Then } |E_t| \leq 10^{-3} \Rightarrow \frac{1}{12} \left(\frac{1}{n}\right)^2 (8) \leq 10^{-3} \Rightarrow \frac{1}{3n} \leq 10^{-3} \Rightarrow \frac{3n}{2} \geq 1000 \Rightarrow n^2 \geq 2000 \Rightarrow n \geq 25.82 \Rightarrow n \geq 26.
\]

47. \[
\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}; \sum_{i=0}^{6} m f(x_i) = 12 \Rightarrow T = \left(\frac{\pi}{6}\right) (12) = \pi
\]

\[
\sum_{i=0}^{6} m f(x_i) = 18 \text{ and } \Delta x = \frac{\pi}{3} = \frac{\pi}{18} \Rightarrow S = \left(\frac{\pi}{18}\right) (18) = \pi.
\]

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48. \[
f^{(4)}(x) \leq 3 \Rightarrow M = 3; \Delta x = \frac{2-1}{n} = \frac{1}{n}. \text{ Hence } |E_t| \leq 10^{-5} \Rightarrow \frac{3}{180} \left(\frac{1}{n}\right)^4 (3) \leq 10^{-5} \Rightarrow \frac{1}{n^4} \leq 10^{-5} \Rightarrow n^4 \geq \frac{10^6}{60}\]

\[
\Rightarrow n \geq 6.38 \Rightarrow n \geq 8 \text{ (n must be even)}.
\]

49. \[
y_{av} = \frac{1}{655-0} \int_{0}^{365} [37 \sin (\frac{2\pi}{365} (x - 101)) + 25] \, dx = \frac{1}{655} \left[ -37 (\frac{2\pi}{365} \cos (\frac{2\pi}{365} (x - 101)) + 25x) \right]_{0}^{365}
\]

\[
= \frac{1}{655} \left[ ( -37 (\frac{365}{2\pi} \cos (\frac{2\pi}{365} (365 - 101)) + 25(365)) - ( -37 (\frac{365}{2\pi} \cos (\frac{2\pi}{365} (0 - 101)) + 25(0))) \right]
\]

\[
= -\frac{37}{2\pi} \cos (\frac{2\pi}{365} (264)) + 25 + \frac{37}{2\pi} \cos (\frac{2\pi}{365} (-101)) = -\frac{37}{2\pi} (\cos (\frac{2\pi}{365} (264)) - \cos (\frac{2\pi}{365} (-101))) + 25
\]

\[
\approx -\frac{37}{2\pi} (0.16705 - 0.16705) + 25 = 25^\circ F
\]

50. \[
av(C) = \frac{1}{675-2} \int_{20}^{675} [8.27 + 10^{-5} (26T - 1.87T^2)] \, dT = \frac{1}{655} \left[ 8.27T + \frac{13}{10} T^2 - \frac{0.62333}{10} T^3 \right]_{20}^{675}
\]

\[
\approx \frac{1}{655} [(5582.25 + 59.23125 - 1917.03194) - (165.4 + 0.052 - 0.04987)] \approx 5.434;
\]

\[
8.27 + 10^{-5} (26T - 1.87T^2) = 5.434 \Rightarrow 1.87T^2 - 26T - 283,600 = 0 \Rightarrow T \approx \frac{26 + \sqrt{676 + 4(1.87)(283,600)}}{2 \times 1.87} \approx 396.45^\circ C
\]
51. (a) Each interval is 5 min = \( \frac{1}{12} \) hour.
\[
\frac{1}{12} [2.5 + 2(2.4) + 2(2.3) + \ldots + 2(2.4) + 2.3] = \frac{29}{12} \approx 2.42 \text{ gal}
\]
(b) (60 mph)(\( \frac{1}{12} \) hours/gal) \( \approx \) 24.83 mi/gal

52. Using the Simpson's rule, \( \triangle x = 15 \Rightarrow \frac{\triangle x}{4} = 5 \);
\[
\sum m(x_i) = 1211.8 \Rightarrow \text{Area} \approx (1211.8)(5) = 6059 \text{ ft}^2;
\]
The cost is Area \( \times \) ($2.10/ft^2) \( \approx \) ($2.10/ft^2)(6059 ft^2) = $12,723.90 \Rightarrow \text{the job cannot be done for $11,000.}

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53. \( \int_{0}^{5} \frac{dx}{\sqrt{9-x^2}} = \lim_{b \to 3^-} \int_{0}^{5} \frac{dx}{\sqrt{9-x^2}} = \lim_{b \to 3^-} \left[ \sin^{-1} \left( \frac{x}{3} \right) \right]_0^b = \lim_{b \to 3^-} \sin^{-1} \left( \frac{b}{3} \right) - \sin^{-1} \left( \frac{0}{3} \right) = \pi - 0 = \frac{\pi}{2} \)

54. \( \int_{0}^{1} \ln x \, dx = \lim_{b \to 0^+} \left[ x \ln x - x \right]_0^1 = (1 \cdot \ln 1 - 1) - \lim_{b \to 0^+} [b \ln b - b] = -1 - \lim_{b \to 0^+} \left[ \frac{\ln b}{b} \right] = -1 - \lim_{b \to 0^+} \left( \frac{1}{b} \right) = -1 - 0 = -1 \)

55. \( \int_{1}^{5} \frac{dy}{y^2 - 3} = \int_{1}^{5} \frac{dy}{y^2 - 3} = 2 \int_{0}^{1} \frac{dy}{y^2 - 3} = 2 \cdot 3 \lim_{b \to 0^+} \left[ y^{1/3} \right]_0^1 = 6 \left( 1 - \lim_{b \to 0^+} b^{1/3} \right) = 6 \)

56. \( \int_{2}^{\infty} \frac{d\theta}{(\theta + 1)^{1/3}} = \int_{2}^{\infty} \frac{d\theta}{(\theta + 1)^{1/3}} + \int_{1}^{2} \frac{d\theta}{(\theta + 1)^{1/3}} + \int_{2}^{\infty} \frac{d\theta}{(\theta + 1)^{1/3}} \) converges if each integral converges, but \( \lim_{\theta \to \infty} \frac{\theta^{1/3}}{(\theta + 1)^{1/3}} = 1 \) and \( \int_{2}^{\infty} \frac{d\theta}{(\theta + 1)^{1/3}} \) diverges \( \Rightarrow \int_{2}^{\infty} \frac{d\theta}{(\theta + 1)^{1/3}} \) diverges

57. \( \int_{3}^{\infty} \frac{2 \, du}{u^2 - 2} = \int_{3}^{\infty} \frac{2 \, du}{u^2 - 2} = \lim_{b \to \infty} \left[ \ln \left| \frac{u - \sqrt{2}}{u + \sqrt{2}} \right| \right]_3^b = \lim_{b \to \infty} \left[ \ln \left| \frac{b - \sqrt{2}}{b + \sqrt{2}} \right| - \ln \left| \frac{3 - \sqrt{2}}{3 + \sqrt{2}} \right| \right] - \ln \left| \frac{3 - \sqrt{2}}{3 + \sqrt{2}} \right| = 0 - \ln \left( \frac{1}{3} \right) = \ln 3 \)

58. \( \int_{1}^{\infty} \frac{3x - 1}{4v^2 - v} \, dv = \int_{1}^{\infty} \left( \frac{3x - 1}{4v^2 - v} \right) \, dv = \lim_{b \to \infty} \left[ \ln v - \frac{1}{v} - \ln (4v - 1) \right]_1^b = \lim_{b \to \infty} \left[ \ln \left( \frac{b}{4b^2 - 1} \right) - \ln \left( \frac{1}{b} \right) \right] - (\ln 1 - 1 - \ln 3) = \ln 3 - 1 + \ln 3 = 0 + 1 + \ln 3 = 1 + \ln 3 \)

59. \( \int_{0}^{\infty} x^2e^{-x} \, dx = \lim_{b \to \infty} \left[ -x^2e^{-x} - 2xe^{-x} - 2e^{-x} \right]_0^b = \lim_{b \to \infty} \left( -b^2e^{-b} - 2be^{-b} - 2e^{-b} \right) = 0 + 2 = 2 \)

60. \( \int_{0}^{1} xe^{3x} \, dx = \lim_{b \to \infty} \left[ \frac{1}{3} e^{3x} - \frac{1}{9} e^{3x} \right]_0^b = -\frac{1}{3} - \lim_{b \to \infty} \left( \frac{1}{3} e^{3b} - \frac{1}{9} e^{3b} \right) = -\frac{1}{3} - 0 = -\frac{1}{3} \)

61. \( \int_{-\infty}^{0} \frac{dx}{4x^2 + 5} = 2 \int_{0}^{0} \frac{dx}{4x^2 + 5} = 2 \int_{0}^{0} \frac{dx}{x^2 + \frac{5}{4}} = 2 \lim_{b \to \infty} \left[ \frac{1}{2} \tan^{-1} \left( \frac{2x}{\sqrt{5}} \right) \right]_0^1 = \frac{1}{2} \lim_{b \to \infty} \left[ \frac{1}{2} \tan^{-1} \left( \frac{2x}{\sqrt{5}} \right) - \frac{1}{2} \tan^{-1} (0) \right] = \frac{1}{2} \left( \frac{\pi}{2} - \frac{\pi}{2} \right) = 0 \)

62. \( \int_{-\infty}^{\infty} \frac{4 \, dx}{x^2 + 16} = 2 \int_{-\infty}^{\infty} \frac{4 \, dx}{x^2 + 16} = 2 \lim_{b \to \infty} \left[ \tan^{-1} \left( \frac{x}{4} \right) \right]_0^b = 2 \left( \lim_{b \to \infty} \left[ \tan^{-1} \left( \frac{b}{4} \right) \right] - \tan^{-1} (0) \right) = 2 \left( \frac{\pi}{2} - 0 \right) = \pi \)

63. \( \lim_{b \to \infty} \frac{d}{\sqrt{b^2 + 1}} = 1 \) and \( \int_{0}^{\infty} \frac{d\phi}{\sqrt{\phi^2 + 1}} \) diverges \( \Rightarrow \int_{0}^{\infty} \frac{d\phi}{\sqrt{\phi^2 + 1}} \) diverges
64.  
\[ I = \int_0^\infty e^{-u} \cos u \, du = \lim_{b \to \infty} \left[ -e^{-u} \cos u \right]_0^b = \int_0^\infty e^{-u} \sin u \, du = 1 + \lim_{b \to \infty} \left[ e^{-u} \sin u \right]_0^b = \int_0^\infty (e^{-u}) \cos u \, du \]
\[ \Rightarrow I = 1 + 0 - I \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2} \text{ converges} \]

65.  
\[ \int_1^\infty \frac{\ln z}{z} \, dz = \int_1^\infty \frac{\ln z}{z} \, dz + \int_1^\infty \frac{\ln z}{z} \, dz = \left[ \frac{(\ln z)^2}{2} \right]_1^\infty + \lim_{b \to \infty} \left[ \frac{(\ln z)^2}{2} \right]_1^b = \left( \frac{1}{2} - 0 \right) + \lim_{b \to \infty} \left[ \frac{(\ln b)^2}{2} - \frac{1}{2} \right] = \infty \]
\[ \Rightarrow \text{ diverges} \]

66.  
\[ 0 < \frac{e^t}{\sqrt{t}} \leq e^t \text{ for } t \geq 1 \text{ and } \int_1^\infty e^t \, dt \text{ converges } \Rightarrow \int_1^\infty \frac{e^t}{\sqrt{t}} \, dt \text{ converges} \]

67.  
\[ \int_{-\infty}^{\infty} \frac{2 \, dx}{e^x + e^{-x}} = 2 \int_{-\infty}^{\infty} \frac{2 \, dx}{e^x} \text{ converges } \Rightarrow \int_{-\infty}^{\infty} \frac{2 \, dx}{e^x} \text{ converges} \]

68.  
\[ \int_{-\infty}^{\infty} \frac{dx}{x(1 + e^x)} = \int_{-\infty}^{0} \frac{dx}{x(1 + e^x)} + \int_{0}^{1} \frac{dx}{x(1 + e^x)} + \int_{1}^{\infty} \frac{dx}{x(1 + e^x)} \]
\[ \lim_{x \to 0} \frac{1}{x^2 (1 + e^x)} = \lim_{x \to 0} \frac{x^2 (1 + e^x)}{x^2} = \lim_{x \to 0} (1 + e^x) = 2 \text{ and } \int_{0}^{1} \frac{dx}{x} \text{ diverges } \Rightarrow \int_{0}^{\infty} \frac{dx}{x^2 (1 + e^x)} \text{ diverges} \]

69.  
\[ \int \frac{dx}{\sqrt{x^2 - 1}} = \int \frac{u}{\sqrt{u^2 - 1}} \, du = \int \frac{2u^2 - 2u + 2 - \frac{2}{1 + u}}{1 + u} \, du = \frac{2}{3} u^3 - u^2 + 2u - 2 \ln |1 + u| + C \]
\[ = \frac{2e^2}{3} - x + 2 \sqrt{x - 2} \ln (1 + \sqrt{x}) + C \]

70.  
\[ \int \frac{dx}{4 - x^2} \text{ diverges} \]

71.  
\[ \int \frac{dx}{x(x^2 + 1)^2} = \int \frac{x}{x(x^2 + 1)} \, dx = \int x \, dx - \frac{1}{2} \int \frac{dx}{x + 2} = \frac{x^2}{2} - \frac{1}{2} \ln |x + 2| - \frac{1}{2} \ln |x - 2| + C \]

72.  
\[ \int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{2} \sqrt{1 - x^2}} = \sin^{-1} (x + 1) + C \]

73.  
\[ \int \frac{dx}{x^2 \sin x} = \int 2 \csc^2 x \, dx - \int \frac{\cos x \, dx}{\sin x} + \int \csc x \, dx = -2 \cot x + \frac{1}{\sin x} - \ln |\csc x + \cot x| + C \]
\[ = -2 \cot x + \csc x - \ln |\csc x + \cot x| + C \]

74.  
\[ \int \frac{dx}{\cos x} = \int \sec x \, dx = \int \sec^2 x \, dx - \tan x + C \]

75.  
\[ \int \frac{dx}{x^2 + 1} = \frac{1}{2} \int \frac{dv}{v^2 + 1} + \frac{1}{2} \int \frac{dv}{3 - v} + \frac{1}{2} \int \frac{dv}{3 + v} = \frac{1}{2} \ln \left| \frac{3 + v}{3 - v} \right| + \frac{1}{2} \tan^{-1} \frac{v}{3} + C \]

76.  
\[ \int \frac{dx}{x(1 - x)^2} = \lim_{b \to \infty} \left[ \frac{1}{2} \right]_1^b = \lim_{b \to \infty} \left[ \frac{1}{2} - (-1) \right] = 0 + 1 = 1 \]

77.  
\[ \cos (2\theta + 1) \]
\[ \theta \quad (+) \quad \frac{1}{2} \sin (2\theta + 1) \]
\[ 1 \quad (--) \quad -\frac{1}{2} \cos (2\theta + 1) \]
\[ 0 \quad \Rightarrow \int \theta \cos (2\theta + 1) \, d\theta = \frac{1}{2} \sin (2\theta + 1) + \frac{1}{4} \cos (2\theta + 1) + C \]
78. \[
\int \frac{x^2 \, dx}{x^2 - 2x + 1} = \int (x + 2 + \frac{3x - 2}{x^2 - 2x + 1}) \, dx = \int (x + 2) \, dx + 3 \int \frac{dx}{x^2 - 1} + \int \frac{dx}{x - 1} \\
= \frac{x^2}{2} + 2x + 3 \ln |x - 1| - \frac{1}{x - 1} + C
\]

79. \[
\int \frac{\sin 2\theta \, d\theta}{(1 + \cos 2\theta)} = -\frac{1}{2} \int \frac{d(1 + \cos 2\theta)}{(1 + \cos 2\theta)^2} = -\frac{1}{2(1 + \cos 2\theta)} + C = \frac{1}{2} \sec^2 \theta + C
\]

80. \[
\int_{\pi/4}^{\pi/2} \sqrt{1 + \cos 4x} \, dx = -\frac{\sqrt{2}}{2} \int_{\pi/4}^{\pi/2} \cos 2x \, dx = \left[-\frac{\sqrt{2}}{2} \sin 2x \right]_{\pi/4}^{\pi/2} = \sqrt{2}
\]

81. \[
\int \frac{2x^3 \ln x \, dx}{x + 1} + C = \frac{2x^2}{2} \ln x - \int \frac{4x^2 \, dx}{x + 1} + C = 2x^2 \ln x - 4(2x + 2) + C
\]

82. \[
\int \frac{1}{x^2 - 2x + 2} \, dx \quad \left[ y = 2 - x \right] \rightarrow -\int (2 - y) \frac{dy}{\sqrt{2}} = \frac{\sqrt{2}}{3} y^{3/2} - 4y^{1/2} + C = \frac{\sqrt{2}}{3} (2 - x)^{3/2} - 4(2 - x)^{1/2} + C
\]

83. \[
\int \frac{1}{y - 2y + 2} \, dy = \int (y - 1)^{-1} \, dy = \tan^{-1} (y - 1) + C
\]

84. \[
\int \frac{x \, dx}{\sqrt{8 - 2x - x^2}} = \frac{1}{2} \int \frac{d(2x^2 + 1)}{\sqrt{9 - (x^2 + 1)^2}} = \frac{1}{2} \sin^{-1} \left( \frac{x^2 + 1}{3} \right) + C
\]

85. \[
\int \frac{z^2 + 1}{z^2 + 4} \, dz = \frac{1}{2} \int \left( \frac{1}{z} + \frac{1}{z^2 + 4} \right) \, dz = \frac{1}{2} \ln |z| - \frac{1}{4z} - \frac{1}{8} \ln (z^2 + 4) - \frac{1}{8} \tan^{-1} \frac{z}{2} + C
\]

86. \[
\int x^3 e^{x^2} \, dx = \frac{1}{2} \int x^2 e^{x^2} \, d(x^2) = \frac{1}{2} \left( x^2 e^{x^2} - e^{x^2} \right) + C = \frac{(x^2 - 1)e^{x^2}}{2} + C
\]

87. \[
\int \frac{t \, dt}{\sqrt{9 - 4t^2}} = -\frac{1}{8} \int \frac{d(4 - 4t^2)}{\sqrt{9 - 4t^2}} = -\frac{1}{2} \sqrt{9 - 4t^2} + C
\]

88. \[
u = \tan^{-1} x, \quad du = \frac{dx}{1 + x^2} \quad \text{and} \quad dv = \frac{dx}{x^2}, \quad v = -\frac{1}{x} ;
\]
\[
\int \frac{\tan^{-1} x \, dx}{x} = -\frac{1}{2} \tan^{-1} x + \int \frac{dx}{x(1 + x^2)} = -\frac{1}{2} \tan^{-1} x + \int \frac{dx}{x} - \int \frac{1 \, dx}{x + 1} = -\frac{1}{2} \tan^{-1} x + \ln |x| - \frac{1}{2} \ln (1 + x^2) + C = -\frac{\ln 1 + x^2}{x} + \ln |x| - \ln \sqrt{1 + x^2} + C
\]

89. \[
\int \frac{e^t \, dt}{e^t + 2e^t + 2} = \left[ e^t = x \right] \rightarrow \int \frac{dx}{x + 1} - \int \frac{dx}{x + 2} = \ln |x + 1| - \ln |x + 2| + C = \ln \left| \frac{x + 1}{x + 2} \right| + C
\]

90. \[
\int \tan^3 t \, dt = \int \left( \tan t \sec^2 t - \tan t \right) \, dt = \tan^2 t - \frac{\tan t}{2} + C
\]

91. \[
\int_1^\infty \ln y \, dy = \lim_{b \to \infty} \left[ \frac{x \ln y}{y} \right]_1^b = \lim_{b \to \infty} \left[ \ln x \right]_1^b = \lim_{b \to \infty} \left[ \frac{b}{2} e^b \right] - \left[ \ln x \right]_1^b = \frac{1}{2}
\]

92. \[
\int \frac{\cos v \, dv}{\ln (\sin v)} = \int \frac{\cos v \, dv}{\sin v \ln (\sin v)} \quad \left[ u = \ln (\sin v) \right] \rightarrow \int \frac{du}{u} = \ln |u| + C = \ln |\ln (\sin v)| + C
\]
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93. \[ \int e^{\sqrt{x}} \, dx = \int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + C \]

94. \[ \int e^{\sqrt{3 + 4e^x}} \, dx = \int \frac{u}{u} \, du = \frac{4e^u}{4} \cdot \frac{(3 + u)^{3/2}}{3} + C = \frac{1}{6} (3 + 4e^u)^{3/2} + C \]

95. \[ \int \frac{\sin 5t}{\cos 5t} \, dt = \int \tan^{-1} u \, du = \int \frac{1}{1 + u^2} \, dt = \frac{1}{2} \tan^{-1} (\cos 5t) + C \]

96. \[ \int e^x 3e^x \, dx = \int 3u \, du = 3e^x + C \]

97. \[ \int \left( \sec x + \sec \left( \sin \left( \frac{1}{1 + x^2} \right) \right) \right) \, dx = \int \sec x \, dx = \frac{1}{2} x^2 - \frac{1}{2} \ln(1 + x^2) + C \]

98. \[ \int \left( \frac{\tan^{-1} x}{\sec x} \right) \, dx = \int \tan^{-1} x \, dx = \frac{1}{2} \sec x \tan^{-1} x + \frac{1}{2} \sec x - \frac{1}{2} \ln(1 + x^2) + C \]

99. \[ \int \left( \frac{\ln x + \cos x}{\sqrt{x}} \right) \, dx = \int \frac{\ln x \, dx}{\sqrt{x}} + \int \frac{\cos x \, dx}{\sqrt{x}} = \frac{1}{2} \ln x - \frac{1}{2} x + C \]

100. \[ \int \frac{\ln x}{\sqrt{x}} \, dx = \frac{1}{2} \ln x + \frac{1}{2} \sqrt{x} + C \]

101. \[ \int \frac{x^2 + 1}{x^2} \, dx = \int \frac{A + Bx + C}{x^2} \, dx = 1 + x^2 = A(1 - x + x^2) + (Bx + C)(1 + x) \]

102. \[ \int \frac{\ln x + \cos x}{\sqrt{x}} \, dx = \int \frac{\ln x \, dx}{\sqrt{x}} + \int \frac{\cos x \, dx}{\sqrt{x}} = \frac{1}{2} \ln x - \frac{1}{2} x + C \]

103. \[ \int \sqrt{x} \sqrt{1 + \sqrt{x}} \, dx = \int \frac{w = \sqrt{x} \rightarrow w^2 = x}{2w \, dw = dx} \rightarrow \int 2w^2 \sqrt{1 + w} \, dw \]

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104. \( \int \sqrt{1 + \sqrt{1 + x}} \, dx \); \( \int 1 + 1 \cdot x \, dx = 2w \) \( \int \) \( \int w \, dw = dx \) \( \int 2w \sqrt{1 + w} \, dw \);

\[
\frac{1}{2} \sqrt{1 + w} (1 + \sqrt{1 + x}) \, dx = \frac{7}{8} w (1 + w)^{3/2} - \frac{8}{15} (1 + w)^{5/2} + C
\]

105. \( \int \frac{1}{\sqrt{x^2 + x}} \, dx \); \( u = \sqrt{2u} \Rightarrow u^2 = x \) \( \int \frac{u}{2u} \, du = dx \); \( u = \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \) \( du = \sec^2 \theta \, d\theta \), \( \sqrt{1 + u^2} = \sec \theta \)

\[
\int 2 \sec \theta \, d\theta = 2 \ln \left[ \sec \theta + \tan \theta \right] + C = 2 \ln \left[ \sqrt{1 + u^2} + u \right] + C
\]

106. \( \int_{0}^{1/2} \sqrt{1 + \sqrt{1 - x^2}} \, dx \);

\[
\int_{x = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \, dx = \cos \theta \, d\theta, \sqrt{1 - x^2} = \cos \theta, x = 0 = \sin \theta \Rightarrow \theta = 0, x = \frac{1}{2} = \sin \theta \Rightarrow \theta = \frac{\pi}{6}
\]

\[
\int_{0}^{\pi/6} \sqrt{1 + \cos \theta} \sin \theta \, d\theta = \int_{0}^{\pi/6} \frac{\sin \theta \cos \theta}{\sqrt{1 - \cos \theta}} \, d\theta = \int_{0}^{\pi/6} \sin \theta \cos \theta \, d\theta = \lim_{c \to 0^+} \int_{c}^{\pi/6} \frac{\sin \theta \cos \theta}{\sqrt{1 - \cos \theta}} \, d\theta;
\]

\[
\left[ u = \cos \theta, \, du = -\sin \theta \, d\theta, \, dv = \frac{-\sin \theta}{\sqrt{1 - \cos \theta}} \, d\theta, \, v = 2(1 - \cos \theta)^{1/2} \right]
\]

107. \( \int \frac{\ln x}{x + \ln x} \, dx = \int \frac{\ln x}{x + \ln x} \, dx \); \( u = 1 + \ln x \Rightarrow du = \frac{1}{x} \, dx \)

\[
\int \frac{u - 1}{u} \, du = \int du - \int \frac{1}{u} \, du = u - \ln |u| + C
\]

\[
= (1 + \ln x) - \ln |1 + \ln x| + C = \ln x - \ln |1 + \ln x| + C
\]

108. \( \int \frac{1}{\ln x} \ln |\ln x| \, dx \); \( u = \ln (\ln x) \Rightarrow du = \frac{1}{x} \ln x \, dx \)

\[
\int \frac{1}{u} \, du = \ln |u| + C = \ln |\ln (\ln x)| + C
\]

109. \( \int x \ln x \, dx \); \( u = x \ln x \Rightarrow \ln u = \ln x \ln x = (\ln x)^2 \Rightarrow \frac{1}{u} \, du = \frac{2}{x} \ln x \, dx \Rightarrow du = \frac{2x \ln x}{x} \, dx = \frac{2x \ln x}{x} \, dx \)

\[
\Rightarrow \int \frac{\ln x}{x} \, dx = \int \frac{1}{u} \, du = \frac{1}{2} u + C = \frac{1}{2} x \ln x + C
\]

110. \( \int (\ln x)^n \ln x \, dx \); \( u = (\ln x)^n \Rightarrow \ln u = \ln (\ln x)^n = (\ln x) \ln (\ln x) \Rightarrow \frac{1}{u} \, du = \frac{(\ln x) \ln x}{x} \, dx \)

\[
\Rightarrow du = u \left[ \frac{1}{x} + \frac{\ln x}{x} \right] \, dx = (\ln x)^n \left[ \frac{1}{x} + \frac{\ln x}{x} \right] \, dx \Rightarrow \int du = u + C = (\ln x)^n + C
\]
111. \[\int \frac{x}{x^2+1} \, dx = \int \frac{\frac{x}{\sqrt{1-x^2}}}{\sqrt{1-x^2}} \, dx; [x^2 = \sin \theta, 0 \leq \theta < \frac{\pi}{2}, 2x \, dx = \cos \theta \, d\theta, \sqrt{1-x^2} = \cos \theta] \rightarrow \frac{1}{2} \int \frac{\cos \theta}{\sin \theta \cos \theta} \, d\theta = \frac{1}{2} \int \csc \theta \, d\theta = -\frac{1}{2} \ln|\csc \theta + \cot \theta| + C = -\frac{1}{2} \ln \left| \frac{1 + \sqrt{1-x^2}}{x^2} \right| + C\]

112. \[\int \frac{1 - x}{x} \, dx; [u = \sqrt{1-x} \Rightarrow u^2 = 1 - x \Rightarrow 2u \, du = -dx] \rightarrow \int \frac{-2u^2}{u^2-1} \, du = \int \left( \frac{2u^2 - 2}{u^2-1} \right) \, du; \frac{2}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} \Rightarrow 2A(u+1) + B(u-1) = (A+B)u + A - B \Rightarrow A = 1 \Rightarrow B = -1; \int 2 \, du + \int \left( \frac{1}{u-1} - \frac{1}{u+1} \right) \, du = 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1-x} - \frac{1}{2} \ln \left| \frac{\sqrt{1-x} - 1}{\sqrt{1-x} + 1} \right| + C\]

113. (a) \[\int_a^b f(a - x) \, dx; [u = a - x \Rightarrow du = -dx, x = 0 \Rightarrow u = a, x = a \Rightarrow u = 0] \rightarrow -\int_a^0 f(u) \, du = \int_0^a f(u) \, du, \text{ which is the same integral as } \int_0^a f(x) \, dx.\]
(b) \[\int_0^{\pi/2} \sin x \, dx = \int_0^{\pi/2} \sin (\frac{\pi}{2} - x) \, dx = \int_0^{\pi/2} \cos x \, dx = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} \, dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} \, dx = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} \, dx = \frac{\pi}{2}\]

114. \[\int \sin x \, dx = \int \frac{\sin x + \cos x + \sin x - \sin x}{\sin x + \cos x} \, dx = \int \frac{\sin x + \cos x + \sin x - \sin x}{\sin x + \cos x} \, dx = \int \frac{-\cos x + \sin x}{\sin x + \cos x} \, dx + \int \frac{\sin x}{\sin x + \cos x} \, dx = 2 \int_0^{\pi/2} \sin x \, dx = x - \frac{1}{2} \ln|\sin x + \cos x| + C\]

115. \[\int \tan^2 x \, dx = \int \frac{\tan^2 x + 1}{\sec^2 x} \, dx = \int \frac{\tan^2 x + 1}{\sec^2 x} \, dx = \int \frac{\sec^2 x - 1}{\sec^2 x} \, dx = \int \frac{\sec^2 x - 1}{\sec^2 x} \, dx = \int \frac{1}{\sec^2 x + \tan^2 x} \, dx = \int dx = \frac{1}{\sqrt{2}} \tan^{-1} \left( \sqrt{2} \tan x \right) + C\]

116. \[\int \csc^2 x \, dx = \int \frac{1 - 2 \cos x + \cos^2 x}{\sin^2 x} \, dx = \int \frac{1 - 2 \cos x + \cos^2 x}{\sin^2 x} \, dx = \int \frac{2 \cos x}{\sin x} \, dx + \int \frac{\cos^2 x}{\sin x} \, dx = \int \csc x \, dx + \int \cot^2 x \, dx = -\cot x + 2 \csc x + \int (\csc^2 x - 1) \, dx = -2 \cot x + 2 \csc x - x + C\]

CHAPTER 8 ADDITIONAL AND ADVANCED EXERCISES

1. \(u = (\sin^{-1} x)^2, \, du = \frac{2 \sin^{-1} x \, dx}{\sqrt{1-x^2}}; \, dv = dx, \, v = x;\)
\[\int (\sin^{-1} x)^2 \, dx = x (\sin^{-1} x)^2 - \int \frac{2x \sin^{-1} x \, dx}{\sqrt{1-x^2}}; \]
\(u = \sin^{-1} x, \, du = \frac{dx}{\sqrt{1-x^2}}; \, dv = -\frac{2x \, dx}{\sqrt{1-x^2}}, \, v = 2 \sqrt{1-x^2};\)
\[-\int \frac{2x \sin^{-1} x \, dx}{\sqrt{1-x^2}} = 2 (\sin^{-1} x) \sqrt{1-x^2} - \int 2 \, dx = 2 (\sin^{-1} x) \sqrt{1-x^2} - 2x + C; \text{ therefore}\]
\[\int (\sin^{-1} x)^2 \, dx = x (\sin^{-1} x)^2 + 2 (\sin^{-1} x) \sqrt{1-x^2} - 2x + C\]

2. \(\frac{1}{x} = \frac{1}{x}, \frac{1}{x+1}, \frac{1}{x+1}; \frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)};\)

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\[
\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{\delta x} - \frac{1}{2(\delta x+1)} + \frac{1}{2(\delta x+2)} - \frac{1}{2(\delta x+3)},\]
\[
\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} + \frac{1}{6(\delta x+1)} + \frac{1}{4(\delta x+2)} - \frac{1}{6(\delta x+3)} + \frac{1}{24(\delta x+4)},
\]

\Rightarrow \text{ the following pattern:}

\[
\frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} x^k \ln |x+k| + C.
\]

3. \( u = \sin x \), \( du = \frac{dx}{\sqrt{1-x^2}} \); \( dv = dx \), \( v = \frac{x^2}{2} \);

\[
\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \int \frac{x^2 \, dx}{\sqrt{1-x^2}} = \int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \int \frac{\sin^2 \theta \cos \theta \, d\theta}{2 \cos \theta}
\]
\[
= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \sin^2 \theta \, d\theta = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \left( \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) + C = \frac{x^2}{2} \sin^{-1} x + \frac{\sin \theta}{4} - \frac{\theta}{2} + C
\]

\[
= \frac{x^2}{2} \sin^{-1} x + \frac{x}{4} \sqrt{\frac{1-x^2}{-\sin^{-1} x}} + C
\]

4. \( \int \sin^{-1} \sqrt{y} \, dy = \int \frac{z}{\sqrt{z^2-1}} \, dz \); from Exercise 3, \( \int \sin^{-1} z \, dz \)

\[
= \frac{z^2}{2} + \frac{\sqrt{z^2-1}}{2} \sin^{-1} z + C \Rightarrow \int \sin^{-1} \sqrt{y} \, dy = y \sin^{-1} \sqrt{y} + \sqrt{\frac{y}{1-y^2}} - \sin^{-1} \sqrt{y} + C
\]

\[
= y \sin^{-1} \sqrt{y} + \frac{\sqrt{\frac{y}{1-y^2}}}{2} - \sin^{-1} \sqrt{y} + C
\]

5. \( \int \frac{dt}{\sqrt{1-t^2}} \), \( t = \sin \theta \)

\[
\int \frac{\cos \theta \, d\theta}{\sin \theta - \cos \theta} = \int \frac{du}{\sin \theta - 1} \Rightarrow \int \frac{\tan \theta \, d\theta}{\sec^2 \theta} = \int \frac{du}{\sec \theta + 1}
\]
\[
= \frac{1}{2} \int \frac{du}{u} - \frac{1}{3} \int \frac{du}{u^3} - \frac{1}{2} \int \frac{du}{u^{2/3}} = \frac{1}{2} \ln \left| \frac{u-1}{\sqrt{u^2}} \right| - \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \ln \frac{\tan \theta}{\sec \theta - 1} - \frac{1}{2} \theta + C
\]

\[
= \frac{1}{2} \ln \left( t - 1 - t^2 \right) - \frac{1}{2} \sin^{-1} t + C
\]

6. \( \int \frac{1}{x^4 + 4} \, dx = \int \frac{1}{(x^2 + 2)^2 - 4x^2} \, dx = \int \frac{1}{(x^2 + 2x + 2)(x^2 - 2x + 2)} \, dx
\]
\[
= \frac{1}{16} \int \left[ \frac{2x+2}{x^2 + 2x + 2} - \frac{2x-2}{x^2 - 2x + 2} + \frac{2}{x^2 + 2x + 2} \right] \, dx
\]
\[
= \frac{1}{16} \ln \left| \frac{x^2 + 2x + 2}{x^2 - 2x + 2} + \frac{1}{2} \tan^{-1} (x+1) + \tan^{-1} (x-1) \right| + C
\]

7. \( \lim_{x \to \infty} \int_{-1}^{x} \sin t \, dt = x \lim_{x \to \infty} [-\cos t]_{-x}^{x} = x \lim_{x \to \infty} [-\cos x + \cos (-x)] = x \lim_{x \to \infty} (-\cos x + \cos x) = x \lim_{x \to \infty} 0 = 0
\]

8. \( \lim_{x \to 0^+} \int_{0}^{1} \cos \frac{1}{x} \, dt \), \( \lim_{x \to 0^+} \frac{1}{x} \cos \frac{1}{x} = \lim_{x \to 0^+} \frac{\cos \frac{1}{x}}{x} \Rightarrow \lim_{x \to 0^+} \int_{0}^{1} \cos \frac{1}{x} \, dt \) diverges since \( \int_{0}^{1} \frac{dt}{x^2} \) diverges; thus

\[
\lim_{x \to 0^+} \int_{0}^{1} \cos \frac{1}{x} \, dt \text{ is an indeterminate } 0 \cdot \infty \text{ form and we apply l'Hopital's rule:}
\]
\[
\lim_{x \to 0^+} \int_{0}^{1} \cos \frac{1}{x} \, dt = \lim_{x \to 0^+} -\frac{\int_{0}^{1} \cos \frac{1}{x} \, dt}{\frac{d}{dx}} = \lim_{x \to 0^+} -\frac{(\cos \frac{1}{x})}{x^2} = \lim_{x \to 0^+} \cos x = 1
\]

9. \( \lim_{n \to \infty} \sum_{k=1}^{n} \ln \left( \frac{1 + \frac{k}{n}}{\frac{k}{n}} \right) = \lim_{n \to \infty} \sum_{k=1}^{n} \ln \left( 1 + k \left( \frac{1}{n} \right) \right) \left( \frac{1}{n} \right) = \int_{0}^{1} \ln (1+x) \, dx ; \left[ x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 2 \right]
\]
\[
\rightarrow \int_{0}^{1} \ln u \, du = [u \ln u - u]_{1}^{2} = (2 \ln 2 - 2) - (\ln 1 - 1) = 2 \ln 2 - 1 = \ln 4 - 1
\]

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10. \[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \, dx = \left\lfloor \sin^{-1} x \right\rfloor_0^1 = \frac{\pi}{2} \]

11. \[ \frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cos 2x = 2 \cos^2 x; \ L = \int_0^{\pi/4} \sqrt{1 + \left(\sqrt{\cos 2t}\right)^2} \, dt = \sqrt{2} \int_0^{\pi/4} \cos^2 t \, dt \]

12. \[ \frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{1-x^2+4x^2}{(1-x^2)^2} = \frac{1+2x^2+x^4}{(1-x^2)^2} = \left(\frac{1+\sqrt{x}}{1-x^2}\right)^2; \ L = \int_0^{1/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \]

13. \[ V = \int_a^b 2\pi \left(\begin{array}{c} \text{radius} \\ \text{height} \end{array} \right) \, dx = \int_a^b 2\pi xy \, dx \]

14. \[ V = \int_a^b \pi y^2 \, dx = \pi \int_1^4 \frac{25 \, dx}{x^4(5-x)} \]

15. \[ V = \int_a^b 2\pi \left(\begin{array}{c} \text{radius} \\ \text{height} \end{array} \right) \, dx = \int_0^1 2\pi xe^x \, dx \]

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16. \[ V = \int_0^{\ln^2} 2\pi(\ln 2 - x)(e^x - 1) \, dx \]
\[ = 2\pi \int_0^{\ln^2} [(\ln 2)e^x - \ln 2 - xe^x + x] \, dx \]
\[ = 2\pi \left[ (\ln 2)e^x - (\ln 2)x - xe^x + e^x + \frac{x^2}{2} \right]_0^{\ln^2} \]
\[ = 2\pi \left[ 2\ln 2 - (\ln 2)^2 - 2\ln 2 + 2 + \ln \frac{2\pi^2}{2} - 2\pi(\ln 2 + 1) \right] \]
\[ = 2\pi \left[ -\frac{(\ln 2)^2}{2} - \ln 2 + 1 \right] \]

17. (a) \[ V = \int_1^e \pi [1 - (\ln x)^2] \, dx \]
\[ = \pi \left[ x - x(\ln x)^2 \right]_1^e + 2\pi \int_1^e \ln x \, dx \]
\[ = \pi \left[ x - x(\ln x)^2 + 2(x \ln x - x) \right]_1^e \]
\[ = \pi \left[ -x - x(\ln x)^2 + 2x \ln x \right]_1^e \]
\[ = \pi \left[ -e - e + 2e - (-1) \right] = \pi \]

(b) \[ V = \int_1^e \pi(1 - \ln x)^2 \, dx = \pi \int_1^e [1 - 2 \ln x + (\ln x)^2] \, dx \]
\[ = \pi \left[ x - 2(x \ln x - x) + x(\ln x)^2 \right]_1^e - 2\pi \int_1^e \ln x \, dx \]
\[ = \pi \left[ x - 2(x \ln x - x) + x(\ln x)^2 - 2(x \ln x - x) \right]_1^e \]
\[ = \pi \left[ 5x - 4x \ln x + x(\ln x)^2 \right]_1^e \]
\[ = \pi \left[ 5e - 4e + e - 5 \right] = \pi(2e - 5) \]

18. (a) \[ \int_{x=0}^{x=\ln 0} x(\ln x) \, dx = \lim_{x \to 0^+} x\ln x = 0 = f(0) \Rightarrow f \text{ is continuous} \]

(b) \[ V = \pi \int_0^1 (e^y - 1)^2 \, dy = \pi \int_0^1 (e^y - 2e^y + 1) \, dy = \pi \left[ \frac{e^y}{2} - 2e^y + y \right]_0^1 \]
\[ = \pi \left( \frac{e}{2} - 2e + \frac{1}{2} \right) = \pi \left( \frac{e - 4e + 5}{2} \right) \]

19. (a) \[ \lim_{x \to 0^+} x \ln x = 0 \Rightarrow \lim_{x \to 0^+} f(x) = 0 = f(0) \Rightarrow f \text{ is continuous} \]

(b) \[ V = \int_0^{\ln^2} \pi x^2(\ln x)^2 \, dx; \quad u = (\ln x)^2 \]
\[ du = (2 \ln x) \frac{dx}{x}, \quad dv = x^2 \, dx \]
\[ v = \frac{x^3}{3} \]
\[ = \pi \left( \frac{8 \ln 2}{3} \right) \left( \frac{4\pi^2}{3} \right) \left( \frac{8 \ln 2}{3} - \frac{16 \ln 2}{9} + \frac{16}{3} \right) \]

20. \[ V = \int_0^1 \pi (\ln x)^2 \, dx \]
\[ = \pi \left( \lim_{b \to 0^+} [x(\ln x)^2]_b^1 - 2\int_0^1 \ln x \, dx \right) \]
\[ = -2\pi \left( \lim_{b \to 0^+} [x(\ln x - x)]_b^1 \right) = 2\pi \]
21. M = \int_1^e \ln x \, dx = [x \ln x - x]_1^e = (e - e) - (0 - 1) = 1;
M_s = \int_1^e (\ln x) \left(\frac{\ln x}{2}\right) \, dx = \frac{1}{2} \int_1^e (\ln x)^2 \, dx
= \frac{1}{2} \left( [x(\ln x)^2]_1^e - 2 \int_1^e \ln x \, dx \right) = \frac{1}{2} (e - 2);
M_r = \int_1^e x \ln x \, dx = \left[ \frac{x^2 \ln x}{2} \right]_1^e - \frac{1}{2} \int_1^e x \, dx
= \frac{1}{2} \left[ x^2 \ln x - \frac{x^2}{2} \right]^e_1 = \frac{1}{2} \left( \left( e^2 - \frac{e^2}{2} \right) + \frac{1}{2} \right) = \frac{1}{4} (e^2 + 1);
therefore, \sigma = \frac{M_s}{M} = e^\frac{\sigma+1}{4} \text{ and } \gamma = \frac{M_s}{M} = \frac{\sigma-2}{2}.

22. M = \int_0^1 \frac{2 \, dx}{\sqrt{1 - x^2}} = 2 [\sin^{-1} x]_0^1 = \pi;
M_s = \int_0^1 \frac{2a \, dx}{\sqrt{1 - x^2}} = 2 \left[ -\sqrt{1 - x^2} \right]_0^1 = 2;
therefore, \sigma = \frac{M_s}{M} = \frac{\pi}{2} \text{ and } \gamma = 0 \text{ by symmetry}

23. L = \int_1^e \sqrt{1 + \frac{1}{x^2}} \, dx = \int_1^e \frac{\sqrt{x^2 + 1}}{x} \, dx; \quad \left[ \begin{array}{c} x = \tan \theta \\
\frac{dx}{d\theta} = \sec^2 \theta \, d\theta \end{array} \right] \rightarrow L = \int_{\pi/4}^{\arctan e} \frac{\sec \theta \cdot \sec^2 \theta \, d\theta}{\tan \theta}
= \int_{\pi/4}^{\arctan e} \left( \tan \theta \sec \theta + \sec \theta \right) \, d\theta = \left[ \sec \theta - \ln |\sec \theta + \cot \theta| \right]_{\pi/4}^{\arctan e}
= \left( \sqrt{1 + e^2} - \ln \left( \frac{\sqrt{1 + e^2}}{e} + \frac{1}{e} \right) \right) - \left( \sqrt{2} - \ln \left( 1 + \sqrt{2} \right) \right)
= \sqrt{1 + e^2} - \sqrt{2} + \ln \left( 1 + \sqrt{2} \right)

24. y = ln x \Rightarrow 1 + \left( \frac{dy}{dx} \right)^2 = 1 + x^2 \Rightarrow S = 2\pi \int_a^b x \sqrt{1 + x^2} \, dy \Rightarrow S = 2\pi \int_0^1 e^y \sqrt{1 + e^y} \, dy; \quad \left[ \begin{array}{c} u = e^y \\
u \, du = e^y \, dy \end{array} \right] \rightarrow S = 2\pi \int_0^1 \sqrt{1 + u^2} \, du;
\left[ \begin{array}{c} u = \tan \theta \\
\frac{du}{d\theta} = \sec^2 \theta \, d\theta \end{array} \right] \rightarrow 2\pi \int_{\pi/4}^{\arctan u} \sec \theta \cdot \sec^2 \theta \, d\theta
= 2\pi \left( \frac{1}{2} \right) \left[ \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_{\pi/4}^{\arctan e}
= \pi \left[ e\sqrt{1 + e^2} + \ln \left( \frac{e + \sqrt{2}}{e} \right) \right] - \pi \left[ \sqrt{2} + \ln \left( 1 + \sqrt{2} \right) \right]

25. S = 2\pi \int_0^1 f(x) \sqrt{1 + \left( f'(x) \right)^2} \, dx; f(x) = (1 - x^{2/3})^{3/2} \Rightarrow f'(x)^2 + 1 = \frac{1}{x^{2/3}} \Rightarrow S = 2\pi \int_0^1 (1 - x^{2/3})^{3/2} \cdot \frac{dx}{x^{2/3}}
= 4\pi \int_0^1 (1 - x^{2/3})^{3/2} \left( \frac{x}{x^{2/3}} \right) \, dx; \quad \left[ \begin{array}{c} u = x^{2/3} \\
u \, du = \frac{2}{3} \, dx \end{array} \right] \rightarrow 4 \cdot \frac{2}{3} \pi \int_0^1 (1 - u)^{3/2} \, du = -6\pi \int_0^1 (1 - u)^{3/2} \, d(1 - u)
= -6\pi \cdot \frac{2}{3} \left[ (1 - u)^{5/2} \right]_0^1 = \frac{12\pi}{5}

26. y = \int_1^x \sqrt{t} - 1 \, dt \Rightarrow \frac{dy}{dx} = \sqrt{x} - 1 \Rightarrow L = \int_1^{16} \sqrt{1 + \left( \sqrt{x} - 1 \right)^2} \, dx = \int_1^{16} \sqrt{1 + \sqrt{x} - 1} \, dx
= \frac{16}{5} \int_1^{16} \sqrt{\frac{1}{5}x^{1/4}} \, dx = \frac{32}{5} (16)^{5/4} - \frac{4}{5} (1)^{5/4} = \frac{124}{5}

27. \int_1^\infty \left( \frac{ax}{x^2 + 1} - \frac{1}{x} \right) \, dx = \lim_{b \to \infty} \int_1^b \left( \frac{ax}{x^2 + 1} - \frac{1}{x} \right) \, dx = \lim_{b \to \infty} \left[ \frac{a}{2} \ln (x^2 + 1) - \frac{1}{2} \ln x \right]_1^b = \lim_{b \to \infty} \left[ \frac{1}{2} \ln \left( \frac{a + 1}{x} \right)^b \right]
= \lim_{b \to \infty} \frac{1}{2} \left[ \ln \left( \frac{b(a + 1)}{b} \right)^b - \ln 2^b \right]; \lim_{b \to \infty} \left( \frac{b(a + 1)}{b} \right)^b > \lim_{b \to \infty} b^b \text{ if } a > \frac{1}{2} \Rightarrow \text{ the improper integral diverges if } a > \frac{1}{2}; \text{ for } a = \frac{1}{2}, \lim_{b \to \infty} \sqrt{b + 1} = \lim_{b \to \infty} \sqrt{1 + \frac{1}{b}} = 1 \Rightarrow \lim_{b \to \infty} \frac{1}{2} \left[ \ln \left( \frac{b(a + 1)}{b} \right)^b - \ln 2^{1/2} \right]
\[ \frac{1}{2} (\ln 1 - \frac{1}{2} \ln 2) = -\frac{\ln 2}{4}; \text{ if } a < \frac{1}{2}, \quad \lim_{b \to \infty} \frac{(b+1)^n}{b} < \lim_{b \to \infty} \frac{(b+1)^n}{b+1} = \lim_{b \to \infty} (b+1)^{n-1} = 0 \]

\[
\Rightarrow \lim_{b \to \infty} \ln \left(\frac{(b+1)^n}{b}\right) = -\infty \Rightarrow \text{ the improper integral diverges if } a < \frac{1}{2}; \text{ in summary, the improper integral}
\]

\[ \int_{1}^{\infty} \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right) \, dx \text{ converges only when } a = \frac{1}{2} \text{ and has the value } -\frac{\ln 2}{4}. \]

28. \(G(x) = \lim_{b \to \infty} \int_{0}^{b} e^{-u} \, du = \lim_{b \to \infty} \left[-\frac{1}{x} e^{-x}\right]_0^b = \lim_{b \to \infty} \left(1 - \frac{1-e^{-b}}{x}\right) = \frac{1}{x} - 0 = \frac{1}{x} \text{ if } x > 0 \Rightarrow xG(x) = x \left(\frac{1}{x}\right) = 1 \text{ if } x > 0\)

29. \(A = \int_{1}^{\infty} \frac{dx}{x^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1. \text{ Thus, } p \leq 1 \text{ for infinite area. The volume of the solid of revolution}

about the x-axis is \(V = \pi \int_{1}^{\infty} \pi \left(\frac{1}{x}\right)^2 \, dx = \pi \int_{1}^{\infty} \frac{dx}{x^p}\) which converges if \(2p > 1\) and diverges if \(2p \leq 1. \text{ Thus we want}

\[ p > \frac{1}{2} \] for finite volume. In conclusion, the curve \(y = x^{-p}\) gives infinite area and finite volume for values of \(p\) satisfying \(\frac{1}{2} < p \leq 1.\)

30. The area is given by the integral \(A = \int_{0}^{1} \frac{dx}{x^p};\)

\[ p = 1: \quad A = \lim_{b \to \infty} \ln b = -\lim_{b \to \infty} \ln b = -\infty, \text{ diverges}; \]

\[ p > 1: \quad A = \lim_{b \to \infty} \left[\frac{x^{1-p}}{1-p}\right]_1^b = 1 - \lim_{b \to \infty} b^{1-p} = -\infty, \text{ diverges}; \]

\[ p < 1: \quad A = \lim_{b \to \infty} \left[\frac{x^{1-p}}{1-p}\right]_1^b = 1 - \lim_{b \to \infty} b^{1-p} = 1 - 0, \text{ converges; thus, } p \geq 1 \text{ for infinite area}. \]

The volume of the solid of revolution about the x-axis is \(V = \pi \int_{0}^{\infty} \left[R(y)^2\right] \, dy = \pi \int_{1}^{\infty} \frac{dy}{y^p}\) which converges if \(\frac{2}{p} > 1 \Leftrightarrow p < 2\) (see Exercise 29). In conclusion, the curve \(y = x^{-p}\) gives infinite area and finite volume for values of \(p\) satisfying \(1 < p < 2\), as described above.

31. (a) \(\Gamma(1) = \int_{0}^{\infty} e^{-u} \, du = \lim_{b \to \infty} \int_{0}^{b} e^{-u} \, du = \lim_{b \to \infty} [-e^{-b}]_0^b = \lim_{b \to \infty} \left[-\frac{1}{e^b} - (-1)\right] = 0 + 1 = 1\)

(b) \[u = t^p, \quad du = xt^{p-1} \, dt, \quad dv = e^{-t} \, dt, \quad v = -e^{-t}; \quad x = \text{fixed positive real} \]

\[\Rightarrow \Gamma(x+1) = \int_{0}^{\infty} t^{x+p} e^{-t} \, dt = \lim_{b \to \infty} \left[-t^{x+p} e^{-t}\right]_0^b + x \int_{0}^{\infty} t^{x} e^{-t} \, dt = \lim_{b \to \infty} \left(-\frac{b^{x+p} + 0^x e^0}{p}\right) + x\Gamma(x) = x\Gamma(x) \]

(c) \(\Gamma(n+1) = n\Gamma(n) = n!:\)

\[n = 0: \quad \Gamma(0+1) = \Gamma(1) = 0!; \]

\[n = k: \quad \Gamma(k+1) = k! \text{ for some } k > 0; \]

\[n = k + 1: \quad \Gamma(k+1+1) = (k + 1)\Gamma(k+1) \quad \text{from part (b)}\]

\[= (k+1)k! \quad \text{induction hypothesis} \]

\[= (k+1)! \quad \text{definition of factorial} \]

Thus, \(\Gamma(n+1) = n\Gamma(n) = n! \text{ for every positive integer } n.\)

32. (a) \(\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{2\pi \over x} \text{ and } n\Gamma(n) = n! \Rightarrow n! \approx n \left(\frac{n}{e}\right)^n \sqrt{2\pi \over n} = \left(\frac{n}{e}\right)^n \sqrt{2\pi \pi} \)

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33. $e^{2x}$  

\[ (+) \quad \cos 3x \]

\[ 2e^{2x} \rightarrow \frac{1}{3} \sin 3x \]

\[ 4e^{2x} \rightarrow -\frac{1}{9} \cos 3x \]

\[ I = \int e^{2x} \sin 3x + \frac{2e^{2x}}{9} \cos 3x - \frac{1}{9} \int \Rightarrow \frac{13}{9} I = e^{2x} (3 \sin 3x + 2 \cos 3x) \Rightarrow I = \frac{e^{2x}}{11} (3 \sin 3x + 2 \cos 3x) + C \]

34. $e^{3x}$  

\[ (+) \quad \sin 4x \]

\[ 3e^{3x} \rightarrow -\frac{1}{4} \cos 4x \]

\[ 9e^{3x} \rightarrow -\frac{1}{16} \sin 4x \]

\[ I = -\frac{e^{3x}}{4} \cos 4x + \frac{4e^{3x}}{16} \sin 4x - \frac{9}{16} \int \Rightarrow \frac{25}{16} I = e^{3x} (3 \sin 4x - 4 \cos 4x) \Rightarrow I = \frac{e^{3x}}{25} (3 \sin 4x - 4 \cos 4x) + C \]

35. $\sin 3x$  

\[ (+) \quad \sin x \]

\[ 3 \cos 3x \rightarrow -\cos x \]

\[ -9 \sin 3x \rightarrow -\sin x \]

\[ I = -\sin 3x \cos x + 3 \cos 3x \sin x + 9I \Rightarrow -8I = -\sin 3x \cos x + 3 \cos 3x \sin x \Rightarrow I = \frac{\sin 3x \cos x - 3 \cos 3x \sin x}{8} + C \]

36. $\cos 5x$  

\[ (+) \quad \sin 4x \]

\[ -\sin 5x \rightarrow -\frac{1}{4} \cos 4x \]

\[ -25 \cos 5x \rightarrow -\frac{1}{16} \sin 4 \]

\[ I = -\frac{1}{4} \cos 5x \cos 4x - \frac{5}{16} \sin 5x \sin 4x + \frac{25}{16} \int \Rightarrow -\frac{9}{16} I = -\frac{1}{4} \cos 5x \cos 4x - \frac{5}{16} \sin 5x \sin 4x \Rightarrow I = \frac{1}{8} (4 \cos 5x \cos 4x + 5 \sin 5x \sin 4x) + C \]

37. $e^{ax}$  

\[ (+) \quad \sin bx \]

\[ ae^{ax} \rightarrow -\frac{1}{b} \cos bx \]

\[ a^2 e^{ax} \rightarrow -\frac{1}{b^2} \sin bx \]

\[ I = -\frac{e^{ax}}{b} \cos bx + \frac{ae^{ax}}{b} \sin bx - \frac{ae^{ax}}{b} \int \Rightarrow \left( \frac{x^2 + b^2}{b} \right) I \Rightarrow e^{ax} (a \sin bx - b \cos bx) \Rightarrow I = \frac{e^{ax}}{x^2 + b^2} (a \sin bx - b \cos bx) + C \]
38. \( e^{ax} \)  
\[ \frac{d}{dx} e^{ax} = \frac{a}{b} \sin bx \]
\[ a^{2}e^{ax} = \frac{-1}{b} \cos bx \]

\[ I = \frac{e^{ax}}{b} \sin bx + \frac{a}{b^2} \cos bx - \frac{x^2}{b^2} \]
\[ \Rightarrow I = \frac{e^{ax}}{b^2} (a \cos bx + b \sin bx) + C \]

39. \( \ln(ax) \)  
\[ \frac{d}{dx} \ln(ax) = \frac{1}{x} \]
\[ \int \frac{1}{x} dx = \ln(ax) - x + C \]

40. \( \ln(ax) \)  
\[ \frac{d}{dx} \ln(ax) = \frac{1}{x} \]
\[ \int \frac{1}{x} dx = \frac{1}{3} x^3 + C \]

41. \( \int \frac{dx}{1-\sin x} = \int \frac{2 \cos x}{1-(\sin x)^2} dx = \int \frac{2 \cos x}{1-\tan^2(\frac{x}{2})} dx = 2 \int \frac{dx}{1-\tan(\frac{x}{2})} + C \]

42. \( \int \frac{dx}{1-\sin x + \cos x} = \int \frac{2 \tan \left( \frac{x}{2} \right)}{1-\tan^2 \left( \frac{x}{2} \right)} dx = \int 2 \frac{dx}{1-\tan^2 \left( \frac{x}{2} \right)} = 2 \int \frac{dx}{1+z^2} = \ln |1+z| + C \]

43. \( \int_{0}^{\sqrt{2}/2} \frac{dx}{\sin x} = \int_{0}^{1} \frac{2 \tan \left( \frac{x}{2} \right)}{1-(\tan x)^2} dx = \int_{0}^{1} \frac{2 dx}{1-\tan^2 \left( \frac{x}{2} \right)} = \int_{0}^{1} \frac{dx}{1+z^2} = \frac{1}{2} \ln |1+z| + C \]

44. \( \int_{\sqrt{3}/3}^{\sqrt{2}/2} \frac{dx}{\cos x} = \int_{\sqrt{3}/3}^{1} \frac{2 \cos \left( \frac{x}{2} \right)}{1+\cos x} dx = \int_{\sqrt{3}/3}^{1} \frac{2 dx}{1+\cos x} = \ln |1+z| + C \]

45. \( \int_{0}^{\sqrt{2}/2} \frac{dt}{1-\tan \theta} = \int_{0}^{1} \frac{2 \sec^2 \theta}{1-(\tan \theta)^2} dt = \int_{0}^{1} \frac{2 dx}{1+\tan^2 \left( \frac{x}{2} \right)} = \frac{1}{3} \tan^{-1} \frac{1}{\sqrt{3}} + C \]

46. \( \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{\cos \theta d\theta}{\sin \theta \cos \theta + \sin \theta} = \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{\sec \theta}{1+\tan \theta} d\theta = \int_{1}^{\sqrt{3}/2} \frac{2 \sec^2 \theta}{2-\sec^2 \theta} \sec \theta d\theta = \int_{1}^{\sqrt{3}/2} \frac{1-\sec^2 \theta}{2} d\theta \]
\[ = \left[ \frac{1}{2} \ln z - \frac{z^2}{4} \right]_{1}^{\sqrt{3}} = \left( \frac{1}{2} \ln \sqrt{3} - \frac{3}{4} \right) - \left( 0 - \frac{1}{2} \right) = \ln 3 - \frac{1}{2} = \frac{1}{4} (\ln 3 - 2) = \frac{1}{2} \left( \ln \sqrt{3} - \frac{1}{2} \right) \]

47. \( \int \frac{dt}{\sin \left( \frac{1}{2} \theta \right) \cos \left( \frac{1}{2} \theta \right)} = \int \frac{2 \sec \theta}{2-\sec^2 \theta} d\theta = \frac{1}{\sqrt{2}} \ln \left| \frac{z + \sqrt{2}}{z + 1} \right| + C \]

\[ = \frac{1}{\sqrt{2}} \ln \left| \frac{\tan \left( \frac{1}{2} \theta \right) + 1}{\sqrt{2} \tan \left( \frac{1}{2} \theta \right) + 1} \right| + C \]
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48. \( \int \frac{\cos t}{1 - \cos t} \, dt = \int \left( \frac{1 - \cos^2 t}{1 - \cos t} \right) \frac{\, dt}{\sin t} = \int \frac{2 \, dt}{(1 - \cos t)\sin t} = \int \frac{2 \, dx}{(1 + x^2)\sqrt{1 - x^2}} = \int \frac{2 \, dx}{(1 + x^2)(1 + x^2 - 1 + x^2)} \\
= \int \frac{(1 - x^2) \, dx}{(1 + x^2)\sqrt{x^2}} = \int \frac{dx}{\sqrt{x^2}} - \int \frac{dx}{1 + x^2} = \int \frac{dx}{\sqrt{x^2}} - 2 \int \frac{dx}{x^2 + 1} = - \frac{1}{2} - 2 \tan^{-1} z + C = - \cot \left( \frac{1}{2} \right) - t + C \\

49. \int \sec \theta \, d\theta = \int \frac{\, d\theta}{\cos \theta} = \int \left( \frac{2 \, dt}{1 + x^2} \right) = \int \frac{2 \, dt}{1 - z} = \int \frac{2 \, dx}{(1 + z)(1 - z)} = \int \frac{dx}{1 + z} + \int \frac{dx}{1 - z} \\
= \ln |1 + z| - \ln |1 - z| + C = \ln \left| \frac{1 + \tan \left( \frac{\theta}{2} \right)}{1 - \tan \left( \frac{\theta}{2} \right)} \right| + C \\

50. \int \csc \theta \, d\theta = \int \frac{\, d\theta}{\sin \theta} = \int \left( \frac{2 \, dt}{1 + x^2} \right) = \int \frac{dx}{z} = \ln |z| + C = \ln \left| \tan \left( \frac{\theta}{2} \right) \right| + C